NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2600

STRESSES AND DEFORMATIONS IN WINGS SUBJECTED TO TORSION

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NACA
Washington
February 1952

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SUMMARY

Basic equations of Kármán and Chien (given in "Torsion with Variable Twist," Jour. Aero. Sci., vol. 13, no. 10, Oct. 1946, pp. 503-510) are solved by representing the shape of a torsion box by means of a Fourier series. The coefficients of the series are determined by conventional methods. Angles of twist, longitudinal stresses, and shear stresses are determined in terms of the series coefficients. The method is applied to the calculation of angles of twist and stresses in torsion boxes of rectangular, elliptical, and airfoil cross section.

Results obtained for angles of twist and normal stresses are in good agreement with results of Kármán and Chien except at sharp corners. Results obtained for shear stresses indicate the necessity for use of a large number of terms of the series for satisfactory accuracy.

INTRODUCTION

Experimental and theoretical investigations related to torsional stresses and deformations are given in references 1 to 18. In reference 1, Kármán and Chien have developed equations applicable to the problem of restrained torsion of tubes of arbitrary constant cross section loaded with a couple. Their solution of these equations was given for rectangles. In general, however, the method of solution of the equations given by them is not easily accomplished. It is the purpose of the present report to give the theory of an approximate method of solution of these equations and the results of the application of this method to tubes of rectangular, elliptical, and airfoil cross sections. This work was conducted at the Oregon State College under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

SYMBOLS

In figures 1 to 5 are shown coordinate systems adopted for this report.

x	coordinate along axis of tube
y'	ordinate of airfoil measured from chord
s	distance along periphery of cross section from some arbitrary point O'
r	normal distance from shear center of section to tangent to wall at s
r_o	distance from leading edge to shear center
t	thickness of wall of tube
σ_x	normal stress in skin in direction parallel to x-axis
σ_s	normal stress in skin in direction perpendicular to σ_x
τ	shear stress in skin in direction of x- and s-axes
ϵ_x	normal strain in skin in direction of x-axis
u	displacement of any point in skin in x-direction
θ	rate of twist of tube per unit length in x-direction
E	Young's modulus
μ	Poisson's ratio
G	shear modulus

$$I = \oint t r^2 \, ds$$

$$I_z = \oint t (y')^2 \, ds$$

$\psi = \int_0^s y' \, ds$

$k^2 = \frac{1 - \mu}{2}$

a, b dimensions of rectangles and ellipses (see figs. 3 and 4)

ξ, η coordinates of point on ellipse (see fig. 4)

$\phi = \sin^{-1} \frac{\xi}{b}$ (for ellipse)

n, m, i positive integers

A_n, B_n, C_n, A_m, B_m, C_m series coefficients

λ_n coefficient

A enclosed area of cross section $\left(\frac{1}{2}\oint r \, ds\right)$

l total length of cross section $\left(\oint ds\right)$

M twisting moment

THEORY FOR TORSION TUBE OF CONSTANT CROSS SECTION

Assumptions

The development of the equations of Kármán and Chien are based on the following assumptions:

(1) Torsion tube is of constant cross section of arbitrary shape.

(2) Tube is loaded with constant torsional couple acting about an axis perpendicular to the cross section.

(3) Bending stiffness of thin walls is negligible.

(4) Displacement of any point on wall of the tube is composed of a displacement due to rigid rotation of cross section in plane of cross section plus a displacement due to warping. The latter displacement is in a direction parallel to the axis of the cylindrical tube.

(5) The torsional deformations can be assumed to be independent of deformations due to bending and shear loads. This implies that the principle of superposition is applicable.

(6) Because of the presence of bulkheads, the strain in the circumferential direction is negligibly small.

General Theory

Basic equations.- On the basis of the above assumptions, Kármán and Chien have shown that the following equations are applicable (reference 1, equations 7, 9, 11, and 12):

$$\frac{2}{1-\mu}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial s^2}+\theta\frac{dr}{ds}=0 \tag{1}$$

$$\oint\left(\frac{\partial u}{\partial s}\right)rt\ ds+I\theta=\frac{M}{G} \tag{2}$$

$$-\oint u\frac{dr}{ds}t\ ds+I\theta=\frac{M}{G} \tag{2a}$$

from which, for constant wall thickness,

$$\frac{1}{k^2}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial s^2}+\frac{t}{I}\frac{dr}{ds}\oint u\frac{dr}{ds}\cdot ds+\frac{M}{GI}\frac{dr}{ds}=0 \tag{3}$$

The general problem is to find a function $u = u(x,s)$ that will satisfy equation (3) and the boundary conditions. In any particular problem $r = r(s)$ is known from the geometry of the cross section. This may, however, not easily be expressed analytically.

An approximate solution.- For the approximate solution, let the function $r = r(s)$ be given by a trigonometric series of the following form:

$$\frac{r}{l} = C_o' + \sum C_n' \cos \frac{2n\pi s}{l} \tag{4}$$

This series is periodic in the interval $s = l$. The coefficients may be determined by a number of methods to be discussed later.

Differentiation of equation (4) with respect to s gives

$$\frac{dr}{ds} = -\sum 2n\pi\, C_n' \sin \frac{2n\pi s}{l} \tag{5}$$

Let $C_n = -2n\pi\, C_n'$. Then equation (5) becomes

$$\frac{dr}{ds} = \sum C_n \sin \frac{2n\pi s}{l} \tag{6}$$

A particular solution of equation (3) may be shown (reference 1) to be

$$u_1(s) = \frac{Ml}{2GAt}\left[\frac{s}{l} - \frac{l^2}{2A}\int_0^{\frac{s}{l}} \left(\frac{r}{l}\right) d\left(\frac{s}{l}\right)\right] \tag{7}$$

The solution of equation (3) may then be written in the form

$$u = u_1(s) + u_2(x,s) \tag{8}$$

If equations (8) and (7) are substituted in equation (3), the following is obtained:

$$\frac{1}{k^2}\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial s^2} + \frac{t}{I}\frac{dr}{ds}\oint u_2 \frac{dr}{ds} \cdot ds = 0 \tag{9}$$

If the condition that the end $x = 0$ is restrained, then at $x = 0$, $u = 0$ and $u_1 = -u_2$.

If at the other end, say $x = \infty$, the end is unrestrained, then $\sigma_x = 0$, or since

$$\sigma_x = \frac{E}{1 - \mu^2} \qquad \epsilon_x = \frac{E}{1 - \mu^2}\frac{\partial u}{\partial x}$$

then at $x = \infty$

$$\frac{\partial u}{\partial x} = \frac{\partial u_2}{\partial x} = 0$$

A function u_2 satisfying these boundary conditions may be written as

$$u_2 = -\sum A_n e^{-\lambda_n kx} \sin \frac{2n\pi s}{l} \tag{10}$$

The coefficients A_n may be determined by the condition that at $x = 0$, $u_1 = -u_2$, or

$$\sum A_n \sin \frac{2n\pi s}{l} = \frac{Ml}{2GAt}\left[\frac{s}{l} - \frac{l^2}{2A}\int_0^{\frac{s}{l}} \left(\frac{r}{l}\right) d\left(\frac{s}{l}\right)\right] \tag{11}$$

Let

$$B_n = \frac{2GAt}{Ml}(A_n) \tag{12}$$

Equation (11) may then be written

$$\sum B_n \sin \frac{2n\pi s}{l} = \frac{s}{l} - \frac{l^2}{2A}\int_0^{\frac{s}{l}} \left(\frac{r}{l}\right) d\left(\frac{s}{l}\right) \tag{13}$$

Equation (13) may be used to determine the coefficients B_n and consequently A_n when the geometry of the cross section is known.

The solution of equation (3) satisfying conditions at $x = 0$ and $x = \infty$ may then be written as

$$u = u_1(s) + u_2(x,s) = \sum A_n \sin\frac{2n\pi s}{l} - \sum A_n e^{-\lambda_n kx} \sin\frac{2n\pi s}{l}$$

or

$$u = \sum A_n\left(1 - e^{-\lambda_n kx}\right) \sin\frac{2n\pi s}{l} \tag{14}$$

Since equation (14) satisfies the condition at $x = 0$ and $x = \infty$, it then is necessary to determine whether this is a solution that is applicable elsewhere.

The particular solution u_1 satisfies equation (3) so that it remains to determine whether u_2 satisfies equation (9). Now from equation (10)

$$\frac{1}{k^2}\frac{\partial^2 u_2}{\partial x^2} = -\sum \lambda_n{}^2 A_n e^{-\lambda_n kx} \sin\frac{2n\pi s}{l}$$

and

$$\frac{\partial^2 u_2}{\partial s^2} = \sum \frac{4\pi^2 n^2}{l^2} A_n e^{-\lambda_n kx} \sin\frac{2n\pi s}{l}$$

and equation (9) may, by use of the above relations and equation (10), be written

$$-\sum \lambda_n{}^2 A_n e^{-\lambda_n kx} \sin\frac{2n\pi s}{l} + \sum \frac{4\pi^2 n^2}{l^2} A_n e^{-\lambda_n kx} \sin\frac{2n\pi s}{l} -$$

$$\frac{t}{I}\frac{dr}{ds}\oint\left(\sum A_n e^{-\lambda_n kx} \sin\frac{2n\pi s}{l}\right)\frac{dr}{ds} \cdot ds = 0 \tag{15}$$

Since $dr'/ds = \sum C_n \sin 2n\pi s/l$, the integral in the above equation may be evaluated. Since

$$\oint\left(\sum A_n e^{-\lambda_n kx} \sin \frac{2n\pi s}{l}\right)\left(\sum C_n \sin \frac{2n\pi s}{l}\right) ds = \frac{l}{2} \sum A_m C_m e^{-\lambda_m kx}$$

then equation (15) may be written

$$-\sum \lambda_n{}^2 A_n e^{-\lambda_n kx} \sin \frac{2n\pi s}{l} + \sum \frac{4\pi^2 n^2}{l^2} A_n e^{-\lambda_n kx} \sin \frac{2n\pi s}{l} -$$

$$\frac{tl}{2I}\left(\sum C_n \sin \frac{2n\pi s}{l}\right)\left(\sum A_m C_m e^{-\lambda_m kx}\right) = 0 \tag{16}$$

Equation (16) would be satisfied if every term of the summation is zero. Let any one term be denoted by the subscript i. Then

$$-\lambda_i{}^2 A_i e^{-\lambda_i kx} \sin \frac{2\pi i s}{l} + \frac{4\pi^2 i^2}{l^2} A_i e^{-\lambda_i kx} \sin \frac{2\pi i s}{l} -$$

$$\frac{t}{I} C_i \sin \frac{2\pi i s}{l}\left(\frac{l}{2} \sum A_m C_m e^{\lambda_m kx}\right) = 0$$

This may be solved for λ_i as follows:

$$\lambda_i{}^2 = \left(\frac{2\pi i}{l}\right)^2 - \frac{tl}{2I} \cdot C_i \sum \frac{A_m C_m e^{-\left(\lambda_m - \lambda_i\right)kx}}{A_i} \tag{17}$$

In the development of the solution the λ_i's were treated as constants. However, equation (17) shows that if the λ_i's are constant,

the basic equations are not satisfied exactly. The question then arises as to whether a value of λ_i for use in equation (10) can give a solution that approximates an exact solution to the desired degree of accuracy. Since any constant value of each λ_i will satisfy the conditions at $x = 0$ and at $x = \infty$, it would appear that a constant λ_i can be found that will give reasonably accurate solutions in the region $x > 0$ to $x \leqq \infty$.

As x approaches zero, the value of λ_i obtained from equation (17) approaches

$$\lambda_{i_o} = \sqrt{\left(\frac{2i\pi}{l}\right)^2 - \frac{tl}{2I}\frac{C_i}{A_i}\sum A_m C_m} \tag{18}$$

Other values of λ_i were tried and were denoted by λ_{i_∞} to distinguish from the above. These were taken as

$$\lambda_{i_\infty} = \sqrt{\left(\frac{2i\pi}{l}\right)^2 - \frac{tl}{2I} C_i{}^2} \tag{19}$$

In the subsequent calculations to be discussed later, both of these sets of values were computed and used to compute the angle of twist and stresses. Good agreement with an exact solution was obtained for rectangular sections by use of λ_i obtained from equation (18).

If satisfactory values for λ_i are obtained, then, since the A_i's are obtainable from equations (13) and (11), the displacement function u (equation (14)) is obtainable.

The stresses σ_x, σ_s, and τ and the angle of twist θ may be given in terms of the displacement function as follows (reference 1):

$$\sigma_x = \frac{E}{1 - \mu^2}\frac{\partial u}{\partial x} \tag{20}$$

$$\sigma_s = \mu\sigma_x \tag{21}$$

$$\theta = \frac{M}{IG} + \frac{t}{I} \oint u \frac{dr}{ds} \cdot ds \tag{22}$$

$$\tau = G\left(\frac{\partial u}{\partial s} + r\theta\right) \tag{23}$$

If equation (14) is taken to represent the displacement, then the above equations become

$$\sigma_x = \frac{Ek}{1 - \mu^2} \sum \lambda_n A_n e^{-\lambda_n kx} \sin \frac{2n\pi s}{l} \tag{20a}$$

$$\sigma_s = \frac{Ek\mu}{1 - \mu^2} \sum \lambda_n A_n e^{-\lambda_n kx} \sin \frac{2n\pi s}{l} \tag{21a}$$

$$\theta = \frac{M}{IG} + \frac{t}{I} \oint \left[\sum A_n\left(1 - e^{-\lambda_n kx}\right) \sin \frac{2n\pi s}{l}\right] \times \left(\sum C_n \sin \frac{2n\pi s}{l}\right) ds \tag{22a}$$

$$\tau = G\left[\frac{2\pi}{l} \sum nA_n\left(1 - e^{-\lambda_n kx}\right) \cos \frac{2n\pi s}{l} + r\theta\right] \tag{23a}$$

Assuming equation (14) is a satisfactory approximation of the displacement function, then it is apparent that the stresses and angle of twist may be expressed in terms of the coefficients A_n, C_n, and λ_n which are obtainable from the geometry of the given cross section. The problem is then resolved to that of determination of these coefficients. There are several methods available for determining these.

Rectangular Cross Sections

In figure 3 is shown the notation adopted for rectangular cross sections. The shape of the rectangle may be expressed nondimensionally by the parameter

$$\beta = \frac{a}{a + b} \tag{24}$$

The coefficients C_n' for $n \neq 0$ (equation 4) may be obtained by evaluation of the integrals

$$C_n' = 2\int_0^1 \frac{r}{l} \cos \frac{2n\pi s}{l}\, d\left(\frac{s}{l}\right) \tag{25}$$

The length of the periphery of a rectangle is

$$l = 4(a + b) \tag{26}$$

The values of r are

$$\begin{array}{lr}
0 \leqq s \leqq a & r = b \\
a \leqq s \leqq a + 2b & r = a \\
a + 2b \leqq s \leqq 3a + 2b & r = b \\
3a + 2b \leqq s \leqq 3a + 4b & r = a \\
3a + 4b \leqq s \leqq 4a + 4b & r = b
\end{array}$$

The ratio r/l may be expressed in terms of β as follows:

$$\left.\begin{array}{ll}
\frac{r}{l} = \frac{b}{l} = \frac{1}{4}(1 - \beta) & \frac{s}{l} \geqq 0 \leqq \frac{\beta}{4} \\
\frac{r}{l} = \frac{a}{l} = \frac{\beta}{4} & \frac{s}{l} \geqq \frac{\beta}{4} \leqq \frac{1}{2} - \frac{\beta}{4} \\
\frac{r}{l} = \frac{b}{l} = \frac{1}{4}(1 - \beta) & \frac{s}{l} \geqq \frac{1}{2} - \frac{\beta}{4} \leqq \frac{1}{2} + \frac{\beta}{4} \\
\frac{r}{l} = \frac{a}{l} = \frac{\beta}{4} & \frac{s}{l} \geqq \frac{1}{2} + \frac{\beta}{4} \leqq 1 - \frac{\beta}{4} \\
\frac{r}{l} = \frac{b}{l} = \frac{1}{4}(1 - \beta) & \frac{s}{l} \geqq 1 - \frac{\beta}{4} \leqq 1.00
\end{array}\right\} \tag{27}$$

The integral of equation (25) may then be evaluated to give

$$C_n' = \frac{1 - 2\beta}{n\pi} \sin \frac{n\pi\beta}{2} \tag{28}$$

when n is even. When n is odd, $C_n' = 0$.

The coefficients B_n of equation (13) may be obtained by evaluation of the integrals

$$B_n = 2\int_0^1 \left[\frac{s}{l} - \frac{l^2}{2A}\int_0^{\frac{s}{l}} \left(\frac{r}{l}\right) d\left(\frac{s}{l}\right)\right] \sin\frac{2n\pi s}{l}\, d\left(\frac{s}{l}\right) \tag{29}$$

These coefficients are then

$$B_n = \frac{(2\beta - 1)}{(n\pi)^2\left(\beta - \beta^2\right)} \sin\frac{n\pi\beta}{2} \tag{30}$$

By use of the relation $C_n = -2n\pi C_n'$,

$$C_n = -2n\pi C_n' = -2(1 - 2\beta) \sin\frac{n\pi\beta}{2} \tag{31}$$

The angle of twist may now be determined. For a tube of circular cross section of radius b and wall thickness t_o, the angle of twist is given by

$$\theta_{circle} = \frac{M}{I_o G}$$

where

$$I_o = t_o \oint b^2\, ds = 2\pi b^3 t_o$$

If the angle of twist of a rectangle is divided by the angle of twist of a circle whose diameter is $2b$ and whose skin cross-section area is equal to the skin cross-section area of a rectangle,

$$\frac{\theta}{\theta_{circle}} = \frac{\theta}{M/I_o G}$$

$$= \frac{\frac{M}{IG} + \frac{t}{I}\oint\left[\sum A_n\left(1 - e^{-\lambda_n kx}\right)\sin\frac{2n\pi s}{l}\right]\left(\sum C_n \sin\frac{2n\pi s}{l}\right) ds}{M/I_o G}$$

and from equation (12)

$$\frac{\theta}{M/I_oG} = \frac{I_o}{I} + \frac{l^2 I_o}{2AI} \oint \left[\sum B_n\left(1 - e^{-\lambda_n kx}\right) \sin \frac{2n\pi s}{l}\right]\left[\sum C_n \sin \frac{2n\pi s}{l}\right] d\left(\frac{s}{l}\right)$$

Also, since

$$I_o = 2\pi b^3 t_o$$

$$l^2 = 16(a + b)^2$$

$$A = 4ab$$

$$I = 4abt(a + b)$$

$$2\pi bt_o = 4(a + b)t$$

the above equation becomes

$$\frac{\theta}{M/I_oG} = \frac{b}{a}\left\{1 + \frac{2(a + b)^2}{ab} \oint \left[\sum B_n\left(1 - e^{-\lambda_n kx}\right) \sin \frac{2n\pi s}{l}\right]\left[\sum C_n \sin \frac{2n\pi s}{l}\right] d\left(\frac{s}{l}\right)\right\}$$

The integral in the expression may be determined since terms only of the form $\int_0^1 \sin^2 \frac{2n\pi s}{l}\, d\left(\frac{s}{l}\right)$ have values other than zero. After integration the angle of twist may be written

$$\frac{\theta}{M/I_oG} = \frac{b}{a}\left[1 + \frac{(a + b)^2}{ab} \sum B_n C_n\left(1 - e^{-\lambda_n kx}\right)\right]$$

By use of equation (24) this may be written

$$\frac{\theta}{M/I_oG} = \frac{1 - \beta}{\beta}\left[1 + \frac{1}{\beta(1 - \beta)} \sum \left(1 - e^{-(2\lambda_n b)\frac{kx}{2b}}\right) B_n C_n\right] \qquad (32)$$

In a similar fashion it may be shown that the normal stress is

$$\sigma_x = \frac{E}{1 - \mu^2} \frac{Mlk}{4AbtG} \sum \left(2\lambda_n b\right) B_n e^{-\left(2\lambda_n b\right)\frac{kx}{2b}} \sin \frac{2n\pi s}{l} \tag{33}$$

If it is assumed that $A_o = \pi b^2$, then the above equation becomes

$$\sigma_x = \frac{Ek}{1 - \mu^2} \frac{M}{A_o t_o G} \frac{A_o t_o l}{4Atb} \sum \left(2\lambda_n b\right) B_n e^{-\left(2\lambda_n b\right)\frac{kx}{2b}} \sin \frac{2n\pi s}{l}$$

A nondimensional stress σ_x' may be defined as

$$\sigma_x' = \sigma_x \frac{\left(1 - \mu^2\right) A_o t_o G}{EkM} \tag{34}$$

For the rectangle with $4(a + b)t = 2\pi b t_o$, equation (34) becomes

$$\sigma_x' = \frac{1}{2\beta(1 - \beta)} \sum \left(2\lambda_n b\right) B_n e^{-\left(2\lambda_n b\right)\frac{kx}{2b}} \sin \frac{2n\pi s}{l} \tag{35}$$

A shear stress may be compared with the shear stress in a circular tube of radius b of the same skin cross-sectional area loaded with the same torsional moment. For the circular tube

$$\tau_o = \frac{M}{2A_o t_o} \tag{36}$$

For the general case, from equation (23),

$$\tau = G\left(\frac{\partial u}{\partial s} + r\theta\right)$$

Using the displacement function, equation (14), and equation (12), equation (23) may be written

$$\tau = G\left[\frac{\pi M}{AtG}\sum nB_n\left(1 - e^{-\lambda_n kx}\right)\cos\frac{2n\pi s}{l} + r\theta\right]$$

$$= \frac{M}{2A_o t_o}\left[\frac{2\pi A_o t_o}{At}\sum nB_n\left(1 - e^{-\lambda_n kx}\right)\cos\frac{2n\pi s}{l} + \frac{2A_o t_o G l I_o}{I_o M}\left(\frac{r}{l}\right)\theta\right]$$

Then, using equation (36), the following relation may be written

$$\frac{\tau}{\tau_o} = \left\{\frac{\pi}{\beta}\sum nB_n\left[1 - e^{-(2\lambda_n b)\frac{kx}{2b}}\right]\cos\frac{2n\pi s}{l}\right\} + \frac{4}{1 - \beta}\left(\frac{r}{l}\right)\frac{\theta}{M/I_o G} \tag{37}$$

Elliptic Cross Sections

In order to evaluate the series coefficients for elliptic cross sections, it is necessary to express r as a function of s. In figure 4 is shown the coordinate system adopted for the ellipses. With this notation,

$$s = \int_0^1 \sqrt{\frac{b^2 - (k')^2\xi^2}{b^2 - \xi^2}}\, d\xi \tag{38}$$

where

$$k' = \sqrt{\frac{b^2 - a^2}{b^2}} \tag{39}$$

It may also be shown that

$$r = \frac{a}{\sqrt{1 - (k')^2\left(\frac{\xi}{b}\right)^2}} \tag{40}$$

Therefore

$$\frac{dr}{ds} = \frac{(k')^2 a}{b} \frac{\left(\frac{\xi}{b}\right)\sqrt{1 - \left(\frac{\xi}{b}\right)^2}}{\left[1 - (k')^2\left(\frac{\xi}{b}\right)^2\right]} \tag{41}$$

Let

$$\sin \phi = \frac{\xi}{b} \tag{42}$$

Then equations (38), (40), and (41) may be written as

$$s = b\int_0^{\phi} \sqrt{1 - (k')^2 \sin^2 \phi} \cdot d\phi \tag{38a}$$

$$r = \frac{a}{\sqrt{1 - (k')^2 \sin^2 \phi}} \tag{40a}$$

$$\frac{dr}{ds} = \frac{(k')^2 a}{b} \frac{\sin \phi \cos \phi}{\left[1 - (k')^2 \sin^2 \phi\right]^2} \tag{41a}$$

The integral in equation (38a) will be recognized as the elliptic integral of the second kind (reference 2) $E(\alpha,\phi)$. Tables of values of this integral are given in terms of the parameter ϕ and α, where

$$\alpha = \sin^{-1} k' \tag{43}$$

The total length of the periphery of the ellipse l is given by

$$l = 4b\int_0^{\frac{\pi}{2}} \sqrt{1 - (k')^2 \sin^2 \phi} \cdot d\phi \tag{44}$$

Since the parameter α defines the ratio of minor to major axis of an ellipse (equations (43) and (39)), this will be constant for any particular ellipse. This value is chosen first. Values of ϕ are then assumed, from which ξ and η may be immediately computed. Values of s/b are then obtainable from the tabulated values of the elliptic functions. It is then possible to determine s/l, r/l, and dr/ds corresponding to any assumed value of ϕ or ξ.

The coefficients C_n for the ellipses are then determined by the equation

$$C_n = 2\int_0^1 \frac{dr}{ds} \sin\frac{2n\pi s}{l} \cdot d\left(\frac{s}{l}\right) \tag{45}$$

The integrals in equation (45) may be evaluated by several methods.

In order to evaluate the coefficients B_n, equation (13) is used. For the ellipse this may be reduced to

$$\sum B_n \sin\frac{2n\pi s}{l} = \frac{s}{l} - \frac{\phi}{2\pi} \tag{46}$$

These coefficients are then given by

$$B_n = 2\int_0^1 \left(\frac{s}{l} - \frac{\phi}{2\pi}\right) \sin\frac{2n\pi s}{l}\, d\left(\frac{s}{l}\right) \tag{47}$$

For the elliptic thin-walled tube,

$$I = t\int r^2\, ds = 4ba^2 F\left(\frac{\pi}{2},k'\right) \tag{48}$$

where $F\left(\frac{\pi}{2},k'\right)$ is an elliptic integral of the first kind (reference 2).

Also,

$$A = \pi ab \tag{49}$$

and

$$l = 4b\left[E\left(\frac{\pi}{2},k'\right)\right] \tag{50}$$

$$\frac{\theta}{M/I_oG} = \frac{I_o}{I}\left\{1 - \frac{\pi l^2}{2A}\sum\left[1 - e^{-(2\lambda_n b)\frac{kx}{2b}}\right]nB_nC_n'\right\} \tag{58}$$

$$\sigma_x' = \frac{l^2}{8A}\sum\left[(2\lambda_n b)B_n e^{-(2\lambda_n b)\frac{kx}{2b}} \sin\frac{2\pi ns}{l}\right] \tag{59}$$

$$\frac{\tau}{\tau_o} = \left\{\frac{\pi b l}{A}\sum nB_n\left[1 - e^{-(2\lambda_n b)\frac{kx}{2b}}\right]\cos\frac{2\pi ns}{l}\right\} + \left(\frac{l}{b}\right)\left(\frac{r}{l}\right)\left(\frac{\theta}{M/I_oG}\right) \tag{60}$$

RESULTS OF CALCULATIONS

Rectangular Boxes

Calculations for angles of twist and stresses were made for rectangular boxes with cross sections defined by $\beta = 0.05$, 0.10, 0.15, 0.30, and 0.40. Computations of the coefficients C_n and B_n were first made by application of equations (30) and (31). Tabulated values of these are given in tables 1 to 6.

Values of λ_{i_o} and λ_{i_∞} were computed by use of equations (18) and (19) for values of $\beta = 0.05$, 0.10, and 0.15. For $\beta = 0.20$, 0.30, and 0.40, only λ_{i_o} values were computed. These values are listed in the tables in the nondimensional form $(2\lambda_n b)_o$. The angles of twist for rectangles with $\beta = 0.05$, 0.10, and 0.15 are listed in table 7. These were computed from equation (32). Values were obtained based on values of both λ_o and λ_∞. For rectangles with $\beta = 0.20$, 0.30, and 0.40, angles of twist were computed using only λ_o. In table 7 are also given values of $\theta/(M/IG)$ based upon a value of Poisson's ratio of 0.3, giving $k = 0.592$. This is the ratio of the unit angle of twist to that given by the theory which neglects the effect of warping of the cross section.

After this shear center is located, the airfoil is plotted and values of r/l against s/l may be measured, tabulated, and plotted. The coefficients C_o' and C_n' are then determined from the equations

$$C_o' = \int_0^1 \left(\frac{r}{l}\right) d\left(\frac{s}{l}\right) \tag{54}$$

and

$$C_i' = 2\int_0^1 \frac{r}{l} \cos\frac{2\pi is}{l}\, d\left(\frac{s}{l}\right) \tag{55}$$

Equation (13) is used to evaluate the coefficients B_n. Any one coefficient B_i may be written

$$B_i = 2\int_0^1 \left[\left(\frac{s}{l}\right) - \frac{l^2}{2A}\int_0^{\frac{s}{l}} \left(\frac{r}{l}\right) d\left(\frac{s}{l}\right)\right] \sin\frac{2\pi is}{l}\, d\left(\frac{s}{l}\right) \tag{56}$$

Values of λ_i were computed from equation (18). This was put in the form

$$2b\lambda_i = \frac{4\pi bi}{l}\sqrt{1 - \frac{tl^3}{2I}\cdot\frac{C_i'}{iB_i}\sum mB_mC_m'} \tag{57}$$

by use of the relations

$$A_i = \frac{M}{2GAt} B_i$$

and

$$C_i = -2i\pi C_i'$$

From equations (20a), (22a), and (23a) and by use of the relations applicable to the circular tube of same chord and skin cross-section area,

From equations (22a) and (48) to (50), it may be shown that for the elliptic cross section

$$\frac{\theta}{M/I_oG} = \left(\frac{b}{a}\right)^2 \frac{E\left(\frac{\pi}{2},k'\right)}{F\left(\frac{\pi}{2},k'\right)}\left\{1 + \frac{4}{\pi}\,\frac{b}{a}\left[E\left(\frac{\pi}{2},k'\right)\right]^2 \sum \left[1 - e^{-\left(2\lambda_n bk\right)\frac{x}{2b}}\right]B_nC_n\right\} \tag{51}$$

Similarly, from equations (20a), (34), (49), and (50), the normal stress σ_x' can be obtained as

$$\sigma_x' = \left(\frac{b}{a}\right)\left(\frac{2}{\pi}\right)\left[E\left(\frac{\pi}{2},k'\right)\right]^2 \sum 2\lambda_n bB_n e^{-\left(2\lambda_n bk\right)\frac{x}{2b}} \sin\frac{2n\pi s}{l} \tag{52}$$

Also, the shear stresses may be expressed as

$$\frac{\tau}{\tau_o} = 4\left[E\left(\frac{\pi}{2},k'\right)\right]\left\{\frac{b}{a}\sum nB_n\left[1 - e^{-\left(2\lambda_n bk\right)\frac{x}{2b}}\right]\cos\frac{2n\pi s}{l} + \left(\frac{r}{l}\right)\left(\frac{\theta}{M/I_oG}\right)\right\} \tag{53}$$

Airfoil-Shape Cross Section

In order to determine the coefficients C_n' and C_o' of equation (4), the contour of the airfoil box must first be expressed in the form $r/l = f(s/l)$. In this equation r is measured from the shear center of the box. The shear center of the box was determined by the assumption that the σ_x stresses due to bending may be computed by the formula $\sigma_x = Ny'/I_z$ where

N	bending moment
y'	distance from neutral axis to center of skin
I_z	moment of inertia of skin about neutral axis (chord line for symmetrical wing)

Results for angles of twist $\theta/(M/IG)$ are plotted in figure 6(a) for rectangles with $\beta = 0.05$, 0.10, and 0.15. Values plotted were obtained using both values of λ_i obtained from equations (18) and (19). For the rectangle $\beta = 0.10$, the results obtained by Kármán and Chien for the rectangle with $\beta = 0.10$ are plotted for comparison. The plots are of interest from several viewpoints.

(1) The effect of end restraint on angle of twist disappears substantially when $x/2b$ is approximately 1.0, a distance of one chord length from the restrained end. (The chord here is considered the long side of the rectangle.)

(2) Within small differences, the results obtained for the rectangle with $\beta = 0.10$ by Kármán and Chien check with the results obtained using the value of λ obtained from equation (18).

(3) The series used for the computation of $\theta/(M/I_oG)$ or $\theta/(M/IG)$ converge with fair rapidity so that an excessive number of terms need not be taken.

(4) Because the effect of the restraint is confined principally to the region near $x = 0$, it is reasonable that the value of λ_i satisfying the basic equations in this region are the more logical ones to use.

(5) The method of representing the contour of the box by a Fourier series gives good results for the angles of twist of rectangular cross sections.

(6) It should be noted here that the Kármán-Chien values given are not necessarily exact, as these were also obtained by the use of a finite number of terms of an infinite series.

In figure 6(b), results for angles of twist $\theta/(M/IG)$ are plotted for rectangles with $\beta = 0.05$, 0.10, 0.15, 0.20, 0.30, and 0.40. Results for angles of twist $\theta/(M/I_oG)$ for rectangles with the same β values are shown in figure 7. The values plotted in both of these figures were based on λ_o.

The calculations for σ_x' gave values which are listed in table 8. These are plotted in figures 8 to 13. The series for σ_x' given in equation (35) may be shown to be divergent at the corners at $x = 0$. This indicates an infinite stress (or the existence of a concentrated reaction) at the restrained end at the corners. Furthermore, the series converges less rapidly at the corners of the box than elsewhere. This result does not agree with the results shown in figure 5 of reference 1, in which Kármán and Chien show finite maximum stresses at the corners of

rectangular boxes. The series used by Kármán and Chien may also be shown to be divergent at the corners at the restrained end. The result obtained by them was evidently obtained by use of only a finite number of terms in their series. In figures 14 and 15 are shown comparisons of values of σ_x' as obtained in this report with values of σ_x' obtained by Kármán and Chien and plotted in figure 4 of reference 1.

The series for the shear stress (equation (37)) is slowly convergent, particularly near the corners of the rectangles. However, at the restrained end the shear stresses are easily computed. At the restrained end $x/2b = 0$,

$$\frac{\tau}{\tau_o} = \frac{4}{1 - \beta}\left(\frac{r}{l}\right)\left(\frac{\theta}{M/I_oG}\right) \tag{61}$$

For the rectangle, since r has only two values, this may be simplified to give the following:

When $r = b$ and $x/2b = 0$

$$\frac{\tau}{\tau_o} = \frac{1 - \beta}{\beta} \tag{61a}$$

When $r = a$ and $x/2b = 0$

$$\frac{\tau}{\tau_o} = 1.0 \tag{61b}$$

At the unrestrained end $x/2b = \infty$ it may be shown that the shear stress is given by

$$\frac{\tau}{\tau_o} = \frac{1}{2\beta} \tag{62}$$

In table 9 are shown the results obtained from calculations using the first 40 terms of the series. Because of the slow convergence the values are irregular. However, since the maximum shear values at the restrained end and the minimum values at the free end may be determined exactly, this difficulty in obtaining accurate values of shear stresses at intermediate stations was not felt to justify the extension of the computations to a larger number of terms.

It appears from the study of rectangular sections that the existence of sharp corners is a major cause of high normal and shear stresses. This suggests that the possibility of rounding corners is worthy of investigation.

Boxes of Elliptical Cross Sections

Determination of coefficients.- In order to determine the basic series coefficients B_n and C_n, it is necessary to evaluate the integrals of equations (45) and (47). This was done by two methods. The first consisted of assuming values of ϕ (equation (42)), computing values of (s/l) and dr/ds from equations (38a), (41a), and (44), and then evaluating the integrals of equations (45) and (47) by a numerical integration using a trapezoidal rule. The second method of evaluating these integrals consisted of plotting the functions $(dr/ds) \sin (2n\pi s/l)$ and $(s/l - \phi/2\pi) \sin (2n\pi s/l)$ against s/l and integrating graphically by the use of a planimeter. In table 10 are listed values obtained for the coefficients by these two methods for an ellipse with an a/b ratio of 0.0871.

Computation for angles of twist and stresses.- In table 11 are given values of $\theta/(M/I_oG)$ for this ellipse. Columns (1) and (2) were obtained by using coefficients C_n and B_n and λ_{n_o} obtained by calculations. Values given in column (1) were obtained by using the first 10 coefficients. Values given in column (2) were obtained by using the first 20 coefficients. Values in column (3) were obtained by using the first 10 coefficients obtained by the graphical determination.

At the unrestrained end, the simple torsion theory can be applied. This leads to a value of

$$\left(\frac{\theta}{M/I_oG}\right)_\infty = \frac{4}{\pi^2}\left(\frac{b}{a}\right)^2\left[E\left(\frac{\pi}{2},k'\right)\right]^2$$

For the ellipse with $a/b = 0.0871$, this equation gives $\left(\frac{\theta}{M/I_oG}\right)_\infty = 55.00$. A comparison of this value with the values in table 11 shows that a 20-term series, whose coefficients were obtained by calculation, gives a very good approximation at infinity.

For the ellipse with $a/b = 0.1738$, calculations were based on a 20-term series using coefficients obtained by the approximate computational method of determining the integrals in equations (45) and (47). For all other ellipses calculations were based upon a 10-term series.

In tables 12 to 15 are given coefficients C_n, B_n, and $(2\lambda_n b)_o$ that were used in the computation of $\theta/(M/I_oG)$, σ_x', and τ/τ_o for the ellipses investigated.

In table 16 values are given for angles of twist for the various ellipses. These are plotted in figures 16 and 17. Both $\theta/(M/I_oG)$ and $\theta/(M/IG)$ are plotted.

Values of σ_x' were computed using the coefficients given in columns (2), (3), and (4) of table 10 and those given in tables 12 to 15. These σ_x' values are given in table 17 and are plotted in figures 18 to 22.

Values of τ/τ_o are given in table 18.

Box of Airfoil Cross Section

The airfoil was drawn to a large scale. The coordinate system shown in figure 5 was adopted. The shear center was determined to be 36.8 units from the leading edge, the chord being taken as 65.0. The length of the periphery was obtained by summing up small distances Δs along the contour. This length $l = \sum \Delta s = 142.1$. The enclosed area of the cross section was found to be 663.0. The value of $I/t = \oint r^2\,ds$ was obtained by a summation of $(r^2\Delta s)$ over the periphery. This gave $I/t = 17{,}270$. In table 19 are given values of r/l against s/l obtained from measurements on the airfoil.

Coefficients C_o' and C_n' were obtained from equations (54) and (55). The integrals were evaluated by replacing the integrals by summations. In table 20 are given the coefficients obtained. Twenty coefficients were used in the calculations. Values of r/l were then computed at various values of s/l. These are listed in table 19 for comparison with the measured values. These two sets of r/l against s/l values from table 19 are plotted in figure 23. The accuracy of the series is fairly good except near the points of discontinuity of the profile.

The coefficients B_n were determined by equation (56), the integrals being replaced by summations. Values of these coefficients are listed in table 20.

Coefficients $2\lambda_n b$ were computed from equation (57). These are also listed in table 20.

In table 21 and figure 24 are given the results of computation for angles of twist. These were computed from equation (58). Angles of

twist rapidly approach a constant value as the parameter $x/2b$ is increased. The effect of the restraint at the inner end of the box on angles of twist disappears almost entirely in one chord length from the restrained end.

The results of the calculations for the normal stresses σ_x' are given in table 22 and figure 25. The infinite stress at the corners of the cross section at the restrained end is again shown. In a distance of one chord length from the restrained end, however, the effect of the restraint has been reduced so that the normal stresses are negligibly small. The series used in computing σ_x', converges with fair rapidity so that use of 20 coefficients gives good accuracy, except right at the restrained end. Here the series converges less rapidly and is divergent at the sharp corners.

Table 23 gives the computed values of τ/τ_0 for the airfoil section. The series giving these values does not converge rapidly near the restrained end. The values given in this table are based on 20 coefficients in the series. This is insufficient for good accuracy.

CONCLUSIONS

An approximate method for solving the basic equations of Kármán and Chien was determined and applied to the calculation of angles of twist and stresses in torsion boxes of rectangular, elliptical, and airfoil cross section. From the results of the application of this method, the following conclusions may be drawn:

1. The methods given here for the approximate solution of the Kármán-Chien equations are applicable to torsion boxes having one degree of symmetry of the cross section. The series used to obtain angles of twist converge fairly rapidly. The results check well for the rectangular cross sections with those given by Kármán and Chien. The series for σ_x stresses converges less rapidly, particularly at the restrained end. At the corners of the restrained end, the series diverges, indicating the necessity for concentrated loads at these points.

2. The theory gives results which indicate tendencies that might well be considered in design. The infinite stresses at corners at the restrained end suggest the possibility of high stress concentrations at these points. It would be expected that, even with zero circumferential strains, the computed values for angles of twist and stresses would be modified appreciably by a local yielding in the neighborhood of the restrained ends, particularly at sharp corners.

3. The studies also indicate that a wing section composed of approximately square boxes might be of some advantage in reducing angles of twist and decreasing normal stresses due to torsion.

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Corvallis, Ore., March 28, 1950

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TABLE 1.- SERIES COEFFICIENTS FOR RECTANGLE WITH $\beta = 0.05$

n	C_n	B_n	$(2\lambda_n b)_\infty$	$(2\lambda_n b)_o$
2	-0.2813	-0.07507	5.71	2.88
4	-.5559	-.03708	11.43	5.75
6	-.8169	-.02421	17.17	8.62
8	-1.0575	-.01763	22.96	11.50
10	-1.2723	-.01358	28.78	14.37
12	-1.4559	-.01078	34.65	17.25
14	-1.6028	-.00873	40.58	20.12
16	-1.7112	-.00713	46.55	23.00
18	-1.7770	-.00586	52.57	25.87
20	-1.7998	-.00480	58.62	28.74
22	-1.7767	-.00392	64.68	31.62
24	-1.7122	-.00317	70.81	34.49
26	-1.6034	-.00253	76.93	37.37
28	-1.4560	-.00198	83.04	40.24
30	-1.3263	-.00151	89.11	43.12
32	-1.0570	-.00111	95.23	45.99
34	-.8178	-.00075	101.29	48.87
36	-.5562	-.00046	107.33	51.74
38	-.2816	-.00021	113.35	54.61
40	0	0	119.32	57.49
42	.2822	.00017	125.29	60.36
44	.3544	.00031	131.20	63.24
46	.8175	.00041	137.12	66.11
48	1.0581	.00049	143.03	68.98
50	1.2723	.00054	148.94	71.86
52	1.4565	.00057	154.85	74.74
54	1.6040	.00059	160.78	77.61
56	1.7127	.00058	166.71	80.48
58	1.7775	.00056	172.67	83.36
60	1.8011	.00053	178.64	86.23
62	1.7755	.00049	184.61	89.11
64	1.7122	.00045	190.63	91.98
66	1.5653	.00039	196.62	94.86
68	1.4348	.00034	202.64	97.73
70	1.2704	.00028	208.66	100.60
72	1.0581	.00022	214.67	103.48
74	.8179	.00016	220.68	106.35
76	.7875	.00010	226.64	109.23
78	.2792	.00005	232.67	112.10
80	0	0	238.64	114.98

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TABLE 2.- SERIES COEFFICIENTS FOR RECTANGLE WITH $\beta = 0.10$

n	C_n	B_n	$(2\lambda_n b)_\infty$	$(2\lambda_n b)_0$
2	-0.4941	-0.06957	5.25	3.47
4	-.9397	-.03309	10.57	6.94
6	-1.2936	-.02024	16.05	10.41
8	-1.5208	-.01338	21.67	13.88
10	-1.5989	-.00906	27.43	17.35
12	-1.5208	-.00594	33.29	20.81
14	-1.2942	-.00371	39.18	24.28
16	-.9395	-.00207	45.04	27.76
18	-.4940	-.00086	50.85	31.24
20	0	0	56.50	34.68
22	.4946	.00057	62.17	38.18
24	.9405	.00092	67.68	41.62
26	1.2932	.00108	73.29	45.11
28	1.5210	.00109	78.89	48.58
30	1.5976	.00100	84.53	52.05
32	1.5213	.00084	90.21	55.52
34	1.2939	.00063	95.84	58.92
36	.9405	.00041	101.72	62.48
38	.4940	.00019	107.28	65.86
40	0	0	113.00	69.36
42	-.4932	-.00016	118.78	72.92
44	-.5995	-.00028	124.28	76.32
46	-1.2913	-.00035	129.88	79.79
48	-1.5193	-.00037	135.64	83.35
50	-1.5991	-.00036	141.13	86.73
52	-1.5218	-.00031	146.85	90.23
54	-1.2920	-.00025	152.59	93.73
56	-.9390	-.00018	158.25	97.16
58	-.4954	-.00008	164.00	100.66
60	0	0	169.70	104.16
62	.4945	.00007	175.20	107.54
64	.9405	.00013	180.95	111.10
66	1.2932	.00017	186.41	114.47
68	1.4979	.00018	191.88	117.85
70	1.6001	.00018	197.68	121.41
72	1.5238	.00016	203.40	124.91
74	1.2919	.00013	208.92	128.28
76	.9402	.00009	214.96	131.97
78	.4947	.00005	221.00	135.65
80	0	0	226.00	138.72

TABLE 3.- SERIES COEFFICIENTS FOR RECTANGLE WITH $\beta = 0.15$

n	C_n	B_n	$(2\lambda_n b)_\infty$	$(2\lambda_n b)_o$
2	-0.63554	-0.06320	4.89	3.85
4	-1.13291	-.02815	9.97	7.71
6	-1.38248	-.01527	15.32	11.56
8	-1.33086	-.00827	20.88	15.41
10	-.98973	-.00402	26.48	19.27
12	-.43257	-.00119	31.96	23.10
14	.21892	.00044	37.39	27.00
16	.82293	.00128	42.61	30.83
18	1.13266	.00139	47.85	34.65
20	1.39918	.00139	53.19	38.55
22	1.24758	.00102	58.59	42.40
24	.82142	.00057	63.94	46.20
26	.21880	.00013	69.40	50.10
28	-.43257	-.00022	74.74	53.96
30	-.98910	-.00044	79.93	57.75
32	-1.33236	-.00053	85.28	61.65
34	-1.38361	-.00047	90.63	65.51
36	-1.13266	-.00035	95.92	69.30
38	-.63478	-.00018	101.40	73.22
40	0	0	106.90	77.17
42	.63566	.00015	112.08	80.92
44	1.13015	.00023	117.34	84.75
46	1.38085	.00026	122.71	88.65
48	1.32935	.00023	128.12	92.55
50	.98960	.00016	133.26	96.23
52	.43106	.00006	138.80	100.20
54	-.22043	-.00003	144.00	103.95
56	-.82293	-.00011	149.47	107.92
58	-1.24570	-.00015	154.84	111.82
60	-1.39793	-.00015	160.02	115.58
62	-1.13304	-.00011	165.45	119.47
64	-.81992	-.00008	170.97	123.44
66	-.21967	-.00002	176.10	127.13
68	.42497	.00004	181.50	131.02
70	.98910	.00008	186.86	134.92
72	1.33387	.00010	191.94	138.60
74	1.40810	.00010	197.34	142.50
76	1.13115	.00008	202.96	146.55
78	.63679	.00004	208.00	150.16
80	0	0	213.50	154.13

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TABLE 4.- SERIES COEFFICIENTS FOR RECTANGLE WITH $\beta = 0.20$

n	C_n	B_n	$(2\lambda_n b)_o$
2	-0.70562	-0.05591	4.04
4	-1.14145	-.02261	8.07
6	-1.14133	-.01004	12.11
8	-.70587	-.00349	16.16
10	0	0	20.19
12	.70612	.00155	24.23
14	1.14120	.00184	28.27
16	1.14045	.00142	32.30
18	.70537	.00069	36.35
20	0	0	40.38
22	-.70600	-.00046	44.42
24	-1.14095	-.00063	48.46
26	-1.14135	-.00053	52.50
28	-.70160	-.00029	56.51
30	0	0	60.56
32	.70537	.00022	64.61
34	1.14020	.00031	68.65
36	1.14170	.00028	72.69
38	.70637	.00015	76.73
40	0	0	80.77
42	-.70688	-.00013	84.80
44	-1.14120	-.00019	88.84
46	-1.14108	-.00017	92.88
48	-.70537	-.00010	96.92
50	0	0	100.96
52	.70537	.00008	104.99
54	1.14283	.00012	109.04
56	1.13944	.00011	113.07
58	.70298	.00006	117.11
60	0	0	121.15
62	-.70474	-.00006	125.18
64	-1.14145	-.00009	129.23
66	-1.14396	-.00005	133.26
68	-.69425	-.00005	137.31
70	0	0	141.34
72	.70537	.00004	145.38
74	1.13856	.00007	149.42
76	1.14070	.00007	153.45
78	.70537	.00004	157.50
80	0	0	161.53

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TABLE 5.- SERIES COEFFICIENTS FOR RECTANGLE WITH $\beta = 0.30$

n	C_n	B_n	$(2\lambda_n b)_o$
2	-0.64722	-0.03907	4.04
4	-.76088	-.01149	8.07
6	-.24718	-.00166	12.10
8	.47075	.00178	16.14
10	.80007	.00193	20.17
12	.47025	.00079	24.21
14	-.24706	-.00031	28.23
16	-.76164	-.00072	32.27
18	-.64772	-.00049	36.30
20	0	0	40.34
22	.64659	.00032	44.38
24	-.76264	.00032	48.40
26	-.24655	.00009	52.44
28	-.47125	-.00015	56.47
30	-.79882	-.00021	60.51
32	-.47025	-.00011	64.54
34	.24768	.00005	68.57
36	.76189	.00014	72.61
38	.64910	.00011	76.64
40	0	0	80.68
42	-.64885	-.00009	84.71
44	-.76264	-.00010	88.74
46	-.24844	-.00003	92.78
48	.47025	.00005	96.81
50	.80111	.00008	100.84
52	.47025	.00004	104.88
54	-.24756	-.00002	108.91
56	-.75963	-.00006	112.95
58	-.64835	-.00005	116.98
60	0	0	121.01
62	.65023	.00004	125.05
64	.75963	.00005	129.08
66	.24869	.00001	133.12
68	-.46284	-.00002	137.14
70	-.80007	-.00004	141.18
72	-.47025	-.00002	145.22
74	.24630	.00001	149.25
76	.76365	.00003	153.29
78	.64659	.00002	157.31
80	0	0	161.35

TABLE 6.- SERIES COEFFICIENTS FOR RECTANGLE WITH $\beta = 0.40$

n	C_n	B_n	$(2\lambda_n b)_o$
2	-0.38044	-0.02007	3.72
4	-.23412	-.00310	7.45
6	.23512	.00138	11.16
8	.38032	.00126	14.88
10	0	0	18.61
12	-.38057	-.00056	22.33
14	-.23563	-.00025	26.15
16	.23512	.00019	29.77
18	.38094	.00025	33.49
20	0	0	37.21
22	-.37994	-.00016	40.94
24	-.23512	-.00009	44.66
26	.23512	.00007	48.37
28	.38157	.00010	52.10
30	0	0	55.82
32	-.37981	-.00008	59.54
34	-.23487	-.00004	63.27
36	.23512	.00004	66.98
38	.38182	.00006	70.70
40	0	0	74.43
42	-.38245	-.00005	78.15
44	-.23487	-.00002	81.87
46	.23399	.00002	85.59
48	.37981	.00004	89.31
50	0	0	93.03
52	-.37881	-.00003	96.76
54	-.23399	-.00002	100.48
56	.23563	.00002	104.19
58	.38245	.00003	107.92
60	0	0	111.64
62	-.38157	-.00002	115.36
64	-.23311	-.00001	119.08
66	.23211	.00001	122.80
68	.37448	.00002	126.52
70	0	0	130.24
72	-.37981	-.00002	133.97
74	-.23701	-.00001	137.69
76	.23387	.00001	141.40
78	.38208	.00001	145.13
80	0	0	148.85

TABLE 7.- ANGLES OF TWIST $\theta/(M/I_oG)$ AND $\theta/(M/IG)$

FOR RECTANGLES

(a) For rectangles with β = 0.05, 0.10, and 0.15.

$\frac{x}{2b}$	Based on λ_∞		Based on λ_o	
	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$
β = 0.05				
0	19.00	1.00	19.00	1.00
1/8	81.08	4.26	68.11	3.58
1/4	89.50	4.71	81.15	4.27
1/2	94.60	4.98	89.59	4.72
3/4	95.03	5.01	92.48	4.86
1.0	95.42	5.02	93.97	4.94
2.0	95.70	5.04	95.42	5.02
β = 0.10				
0	9.00	1.00	9.00	1.00
1/8	19.25	2.16	17.07	1.90
1/4	21.94	2.43	20.37	2.26
1/2	23.64	2.63	22.73	2.53
3/4	24.17	2.69	23.62	2.62
1.0	24.37	2.71	24.03	2.67
2.0	24.53	2.73	24.48	2.72
β = 0.15				
0	5.67	1.00	5.67	1.00
1/8	8.79	1.55	8.23	1.45
1/4	9.68	1.71	9.30	1.64
1/2	10.48	1.85	10.24	1.81
3/4	10.76	1.90	10.62	1.87
1.0	10.89	1.92	10.80	1.91
2.0	10.98	1.94	10.98	1.94

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TABLE 7.- ANGLES OF TWIST $\theta/(M/I_oG)$ AND $\theta/(M/IG)$

FOR RECTANGLES - Concluded

(b) For rectangles with β = 0.20, 0.30, and 0.40.

$\frac{x}{2b}$	Based on λ_o	
	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$
	β = 0.20	
0	4.00	1.00
1/8	4.98	1.23
1/4	5.42	1.35
1/2	5.85	1.46
3/4	6.03	1.51
1.0	6.12	1.53
2.0	6.21	1.55
	β = 0.30	
0	2.33	1.00
1/8	2.50	1.07
1/4	2.59	1.11
1/2	2.67	1.15
3/4	2.72	1.17
1.0	2.74	1.18
2.0	2.77	1.19
	β = 0.40	
0	1.50	1.00
1/8	1.52	1.01
1/4	1.53	1.02
1/2	1.55	1.03
3/4	1.55	1.03
1.0	1.56	1.04
2.0	1.56	1.04

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TABLE 8.- NORMAL STRESSES σ_x' FOR RECTANGLES

BASED ON $(2\lambda_n b)_o$

(a) $\beta = 0.05$.

$\frac{x}{2b}$	$\frac{s}{l} = 0.005$	$\frac{s}{l} = 0.01$	$\frac{s}{l} = 0.0125$	$\frac{s}{l} = 0.02$	$\frac{s}{l} = 0.05$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.20$
0	-4.93	-19.76	-----	-10.18	-----	-1.61	-0.42
1/8	-1.99	-3.62	-4.19	-4.94	-3.17	-1.54	-.38
1/4	-.68	-1.28	-1.56	-2.16	-2.36	-1.40	-.35
1/2	-.18	-.35	-.44	-.66	-1.15	-1.01	-.31
3/4	-.08	-.15	-.19	-.29	-.59	-.66	-.24
1.0	-.04	-.08	-----	-.15	-.33	-.43	-.18
2.0	-.01	-.01	-----	-.02	-.05	-.07	-.04

(b) $\beta = 0.10$.

$\frac{x}{2b}$	$\frac{s}{l} = 0.005$	$\frac{s}{l} = 0.01$	$\frac{s}{l} = 0.02$	$\frac{s}{l} = 0.05$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.20$	$\frac{s}{l} = 0.025$
0	-0.69	-2.20	-5.42	-2.30	-0.96	-0.23	-----
1/8	-.51	-1.01	-1.83	-1.84	-.92	-.22	-2.08
1/4	-.23	-.45	-.82	-1.18	-.79	-.21	-.96
1/2	-.07	-.14	-.26	-.50	-.49	-.16	-.31
3/4	-.03	-.06	-.11	-.24	-.30	-.12	-.14
1.0	-.01	-.03	-.06	-.13	-.18	-.08	-----
2.0	0	0	-.01	-.01	-.02	-.01	-----

(c) $\beta = 0.15$.

$\frac{x}{2b}$	$\frac{s}{l} = 0.02$	$\frac{s}{l} = 0.03$	$\frac{s}{l} = 0.0375$	$\frac{s}{l} = 0.05$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.20$
0	-1.34	-2.16	-----	-1.95	-0.70	-0.15
1/8	-.79	-1.12	-1.26	-1.26	-.67	-.14
1/4	-.41	-.57	-.65	-.73	-.56	-.15
1/2	-.14	-.20	-.24	-.29	-.32	-.12
3/4	-.06	-.09	-.11	-.13	-.18	-.08
1.0	-.03	-.04	-----	-.07	-.02	-.01
2.0	0	0	-----	-.01	-.02	-.01

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TABLE 8.- NORMAL STRESSES σ_x' FOR RECTANGLES

BASED ON $\left(2\lambda_n b\right)_o$ - Concluded

(d) $\beta = 0.20$.

$\frac{x}{2b}$	$\frac{s}{l} = 0.02$	$\frac{s}{l} = 0.04$	$\frac{s}{l} = 0.05$	$\frac{s}{l} = 0.06$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.20$
0	-0.54	-1.18	-----	-1.28	-0.58	-0.12
1/8	-.38	-.74	-0.84	-.84	-.53	-.13
1/4	-.23	-.41	-.47	-.50	-.42	-.12
1/2	-.09	-.16	-.19	-.21	-.23	-.09
3/4	-.04	-.07	-----	-.11	-.12	-.06
1.0	-.02	-.04	-----	-.05	-.07	-.04
2.0	0	0	-----	0	-.01	0

(e) $\beta = 0.30$.

$\frac{x}{2b}$	$\frac{s}{l} = 0.03$	$\frac{s}{l} = 0.06$	$\frac{s}{l} = 0.075$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.15$	$\frac{s}{l} = 0.20$
0	-0.18	-0.49	-----	-0.40	-0.18	-0.08
1/8	-.16	-.34	-0.40	-.33	-.17	-.08
1/4	-.12	-.22	-.23	-.24	-.15	-.07
1/2	-.06	-.10	-.11	-.12	-.10	-.05
3/4	-.03	-.05	-.06	-.06	-.06	-.03
1.0	-.01	-.03	-----	-.03	-.03	-.02
2.0	0	0	-----	0	0	0

(f) $\beta = 0.40$.

$\frac{x}{2b}$	$\frac{s}{l} = 0.03$	$\frac{s}{l} = 0.06$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.12$	$\frac{s}{l} = 0.15$	$\frac{s}{l} = 0.20$
0	-0.05	-0.12	-----	-0.16	-0.09	-0.04
1/8	-.04	-.09	-0.16	-.14	-.09	-.04
1/4	-.04	-.07	-.10	-.10	-.07	-.03
1/2	-.02	-.04	-.05	-.05	-.05	-.02
3/4	-.01	-.02	-.03	-.03	-.03	-.02
1.0	-.01	-.01	-----	-.02	-.02	-.01
2.0	0	0	-----	0	0	0

TABLE 9.- SHEAR STRESSES τ/τ_o FOR RECTANGLES

BASED ON $(2\lambda_n b)_o$

(a) $\beta = 0.05$.

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = 0.005$	$\frac{s}{l} = 0.01$	$\frac{s}{l} = 0.02$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.20$	$\frac{s}{l} = 0.25$
0	19.00	19.00	19.00	1.00	1.00	1.00	1.00
1/8	21.11	1.03	19.18	29.32	4.79	4.08	4.05
1/4	17.24	-1.61	19.92	26.34	6.72	5.28	5.26
1/2	15.68	-2.85	19.40	20.77	8.70	6.71	6.59
3/4	15.27	-3.22	19.15	18.19	9.45	7.65	7.51
1.0	15.23	-3.25	19.15	16.89	9.68	8.31	8.19
2.0	14.02	-3.56	18.87	15.30	9.61	9.42	9.42

(b) $\beta = 0.10$.

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = 0.01$	$\frac{s}{l} = 0.02$	$\frac{s}{l} = 0.05$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.20$	$\frac{s}{l} = 0.25$
0	9.00	9.00	9.00	1.00	1.00	1.00	1.00
1/8	8.95	7.58	3.23	2.87	2.35	2.23	1.80
1/4	7.18	6.32	3.29	5.75	3.69	2.77	2.84
1/2	6.19	5.52	3.08	5.66	4.60	3.63	3.60
3/4	5.86	5.26	2.92	5.25	4.87	4.04	4.08
1.0	5.75	5.15	2.84	4.97	4.93	4.35	4.40
2.0	5.62	5.03	2.74	4.61	4.89	4.77	4.88

(c) $\beta = 0.15$.

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = 0.01$	$\frac{s}{l} = 0.02$	$\frac{s}{l} = 0.05$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.20$	$\frac{s}{l} = 0.25$
0	5.67	5.67	5.67	1.00	1.00	1.00	1.00
1/8	5.39	5.07	5.49	3.50	1.98	1.67	1.65
1/4	4.41	4.28	4.91	3.79	2.60	2.07	2.04
1/2	3.87	3.67	4.47	3.52	3.12	2.58	2.54
3/4	3.65	3.46	4.31	3.27	3.25	2.89	2.86
1.0	3.55	3.37	4.23	3.12	3.26	3.07	3.05
2.0	3.47	3.29	4.15	2.95	3.60	3.29	3.20

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TABLE 9.- SHEAR STRESSES τ/τ_o FOR RECTANGLES

BASED ON $(2\lambda_n b)_o$ - Continued

(d) β = 0.20.

$\frac{x}{2b}$	$\frac{s}{l}$ = 0	$\frac{s}{l}$ = 0.01	$\frac{s}{l}$ = 0.02	$\frac{s}{l}$ = 0.05	$\frac{s}{l}$ = 0.10	$\frac{s}{l}$ = 0.20	$\frac{s}{l}$ = 0.25
0	4.00	4.00	4.00	4.00 1.00	1.00	1.00	1.00
1/8	3.80	3.74	3.41	4.71 .98	1.68	1.41	1.38
1/4	3.34	3.28	2.98	4.88 .82	2.15	1.67	1.63
1/2	2.88	2.85	2.83	4.93 .54	2.44	2.01	1.97
3/4	2.70	2.68	2.46	4.90 .38	2.51	2.22	2.18
1.0	2.64	2.62	2.40	4.87 .28	2.50	2.34	2.30
2.0	2.56	2.54	2.33	4.84 .18	2.49	2.48	2.47

(e) β = 0.30.

$\frac{x}{2b}$	$\frac{s}{l}$ = 0	$\frac{s}{l}$ = 0.02	$\frac{s}{l}$ = 0.0625	$\frac{s}{l}$ = 0.10	$\frac{s}{l}$ = 0.1875	$\frac{s}{l}$ = 0.20	$\frac{s}{l}$ = 0.25
0	2.33	2.33	2.33	1.00	1.00	1.00	1.00
1/8	2.02	2.38	2.16	1.55	1.29	1.09	1.08
1/4	1.87	2.24	2.08	1.66	1.42	1.21	1.19
1/2	1.65	2.03	2.00	1.79	1.57	1.37	1.34
3/4	1.56	1.95	1.98	1.80	1.67	1.47	1.44
1.0	1.51	1.90	1.95	1.80	1.72	1.52	1.50
2.0	1.47	1.86	1.93	1.80	1.79	1.60	1.59

TABLE 9.- SHEAR STRESSES τ/τ_o FOR RECTANGLES

BASED ON $(2\lambda_n b)_o$ - Concluded

(f) $\beta = 0.40$.

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = 0.02$	$\frac{s}{l} = 0.0625$	$\frac{s}{l} = 0.10$	$\frac{s}{l} = 0.1875$	$\frac{s}{l} = 0.20$	$\frac{s}{l} = 0.25$
0	1.50	1.50	1.50	1.50 1.00	1.00	1.00	1.00
1/8	1.47	1.46	1.43	1.51 1.00	1.05	1.04	1.04
1/4	1.42	1.41	1.37	1.52 1.00	1.10	1.09	1.09
1/2	1.36	1.30	1.33	1.52 1.00	1.17	1.15	1.15
3/4	1.31	1.30	1.29	1.51 .99	1.20	1.19	1.18
1.0	1.29	1.29	1.28	1.52 1.00	1.23	1.22	1.22
2.0	1.26	1.25	1.26	1.51 .99	1.25	1.24	1.25

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TABLE 10.- SERIES COEFFICIENTS FOR ELLIPSE

WITH a/b = 0.0871

(1)	(2)	(3)	(4)	(5)	(6)	(7)
n	Coefficients by calculation			Coefficients by graph integration		
	C_n	B_n	$(2\lambda_n b)_o$	C_n	B_n	$(2\lambda_n b)_o$
2	0.2501	0.04677	4.94	0.251	0.0471	5.12
4	-.3625	-.01639	10.14	-.384	-.0168	10.05
6	.4270	.00847	14.71	.444	.0086	15.05
8	-.4876	-.00550	19.54	-.512	-.0063	20.67
10	.5338	.00394	24.52	.582	.0036	24.23
12	-.5725	-.00290	29.41	-.595	-.0034	31.31
14	.6032	.00226	34.50	.678	.0020	33.20
16	-.6350	-.00181	39.18	-.614	-.0011	35.80
18	.5868	.00147	44.10	.672	.0010	42.85
20	-.6821	-.00116	48.55	-.749	-.0012	48.94
22	.6992	.00105	53.80	.768	.0015	-----
24	-.7192	-.00092	58.98	-.794	-.0010	-----
26	.7359	.00081	64.07	.794	.0009	-----
28	-.7477	-.00069	68.31	-.838	-.0008	-----
30	.7601	.00058	74.19	.813	.0007	-----
32	-.7669	-.00048	78.94	-.851	-.0006	-----
34	.7745	.00052	83.87	.838	.0005	-----
36	-.7831	-.00047	87.96	-.787	-.0004	-----
38	.7882	.00016	93.99	.870	.0004	-----
40	-.7957	-.00105	97.78	-.857	-.0005	-----

TABLE 11.- ANGLES OF TWIST $\theta/(M/I_oG)$ AND $\theta/(M/IG)$ FOR ELLIPSE

WITH $a/b = 0.0871$

$\frac{x}{2b}$	(1) Obtained by use of first 10 coefficients of table 10, columns (2), (3), and (4)		(2) Obtained by use of first 20 coefficients of table 10, columns (2), (3), and (4)		(3) Obtained by use of first 10 coefficients of table 10, columns (5), (6), and (7)	
	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$
0	34.90	1.00	34.90	1.00	34.90	1.00
1/8	45.02	1.29	47.29	1.36	44.67	1.28
1/4	48.16	1.38	50.54	1.45	48.16	1.38
1/2	50.50	1.45	52.98	1.52	50.61	1.45
3/4	51.23	1.47	53.89	1.54	51.48	1.47
1.0	51.58	1.48	54.27	1.56	51.86	1.49
2.0	52.11	1.49	54.58	1.56	52.14	1.49
∞	52.15	1.49	54.62	1.56	52.18	1.49

TABLE 12.- SERIES COEFFICIENTS FOR

ELLIPSE WITH a/b = 0.1738

n	C_n	B_n	$(2\lambda_n b)_o$
2	0.4378	0.04405	5.22
4	-.5791	-.01457	10.39
6	.6548	.00733	15.56
8	-.6892	-.00435	20.76
10	.7223	.00291	25.86
12	-.7227	-.00201	31.38
14	.7122	.00145	36.40
16	-.6982	-.00109	39.33
18	.6848	.00085	45.38
20	-.6643	-.00068	49.38
22	.6445	.00052	57.44
24	-.6198	-.00044	63.09
26	.5944	.00036	68.46
28	-.5679	-.00028	72.89
30	.5420	.00024	78.34
32	-.5349	-.00020	82.89
34	.4858	.00016	87.93
36	-.4582	-.00012	91.54
38	.4346	.00012	99.71
40	-.4091	-.00008	100.13

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TABLE 13.- SERIES COEFFICIENTS FOR

ELLIPSE WITH $a/b = 0.2588$

n	C_n	B_n	$(2\lambda_n b)_o$
2	0.5404	0.0388	5.31
4	-.6439	-.0114	10.60
6	.6560	.0056	16.02
8	-.6317	-.0028	21.22
10	.5873	.0017	26.58
12	-.5401	-.0011	31.90
14	.4868	.0008	37.52
16	-.4333	-.0004	41.60
18	.3663	.0003	47.48
20	-.3649	-.0002	51.72

TABLE 14.- SERIES COEFFICIENTS FOR

ELLIPSE WITH a/b = 0.5000

n	C_n	B_n	$(2\lambda_n b)_o$
2	0.5287	0.02640	5.04
4	-.3989	-.00498	10.15
6	.2753	.00154	15.10
8	-.1748	-.00054	20.09
10	.0982	.00018	25.22
12	-.0643	-.00009	30.33
14	.0386	.00004	35.41
16	-.0231	-.00002	40.60
18	.0136	.00001	46.60
20	-.0070	0	51.85

TABLE 15.- SERIES COEFFICIENTS FOR

ELLIPSE WITH a/b = 0.8660

n	C_n	B_n	$(2\lambda_n b)_o$
2	0.14184	0.00573	4.28
4	-.02536	-.00024	8.57
6	.00402	.00002	12.85
8	-.00060	0	17.12
10	-.00098	0	21.40
12	.00008	0	25.70
14	-.00005	0	29.90
16	-.00006	0	34.30
18	.00016	0	38.60
20	-.00029	0	42.90

TABLE 16.- ANGLES OF TWIST $\theta/(M/I_oG)$ AND $\theta/(M/IG)$ FOR

ALL ELLIPSES FROM $(2\lambda_n b)_o$

$\frac{x}{2b}$	$\frac{a}{b}$ = 0.0871		$\frac{a}{b}$ = 0.1738		$\frac{a}{b}$ = 0.2588		$\frac{a}{b}$ = 0.5000		$\frac{a}{b}$ = 0.8660	
	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$
0	34.90	1.00	10.97	1.00	5.80	1.00	2.25	1.00	1.16	1.00
1/8	47.29	1.36	12.88	1.17	6.36	1.10	2.34	1.04	1.16	1.00
1/4	50.54	1.45	13.59	1.24	6.61	1.14	2.40	1.07	1.16	1.00
1/2	52.98	1.52	14.16	1.29	6.84	1.18	2.46	1.09	1.16	1.00
3/4	53.89	1.54	14.38	1.31	6.93	1.20	2.49	1.11	1.16	1.00
1.0	54.27	1.56	14.48	1.32	6.96	1.20	2.50	1.11	1.16	1.00
2.0	54.58	1.56	14.55	1.33	6.99	1.21	2.51	1.11	1.16	1.00
∞	54.62	1.56	14.55	1.33	6.99	1.21	2.51	1.11	1.16	1.00

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TABLE 17.- NORMAL STRESSES σ_x' FOR ALL

ELLIPSES FROM $(2\lambda_n b)_o$

(a) $a/b = 0.0871$.

$\frac{x}{2b}$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$
0	0.4806	1.0609	2.5394	3.3680
1/8	.4444	.9727	1.5020	1.5068
1/4	.3778	.7444	.9179	.6832
1/2	.2265	.3872	.3437	.2084
3/4	.1229	.1915	.1503	.0842
1.0	.0624	.0936	.0699	.0385
2.0	.0037	.0052	.0037	.0020

(b) $a/b = 0.1738$.

$\frac{x}{2b}$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$
0	0.3077	0.6261	1.1263	1.5797
1/8	.2505	.4855	.7468	.6844
1/4	.1941	.3728	.4363	.3069
1/2	.1108	.1850	.1632	.0935
3/4	.0564	.0865	.0675	.0377
1.0	.0278	.0409	.0302	.0163
2.0	.0012	.0016	.0012	.0008

(c) $a/b = 0.2588$.

$\frac{x}{2b}$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$
0	0.1992	0.4154	0.7469	1.0597
1/8	.1708	.3339	.4679	.3969
1/4	.1334	.2462	.2748	.1856
1/2	.0731	.1198	.1028	.0597
3/4	.0361	.0549	.0423	.0236
1.0	.0172	.0253	.0185	.0101
2.0	.0008	.0012	.0008	.0005

TABLE 17.- NORMAL STRESSES σ_x' FOR ALL

ELLIPSES FROM $(2\lambda_n b)_o$ - Concluded

(d) a/b = 0.5000.

$\frac{x}{2b}$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$
0	0.1007	0.2031	0.2700	0.2214
1/8	.0804	.1503	.1633	.1088
1/4	.0614	.1072	.1013	.0617
1/2	.0331	.0523	.0420	.0238
3/4	.0166	.0248	.0186	.0102
1.0	.0083	.0119	.0086	.0047
2.0	.0005	.0007	.0005	.0003

(e) a/b = 0.8660.

$\frac{x}{2b}$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$
0	0.0244	0.0383	0.0309	0.0175
1/8	.0184	.0281	.0218	.0122
1/4	.0137	.0205	.0155	.0086
1/2	.0075	.0109	.0079	.0043
3/4	.0040	.0058	.0042	.0023
1.0	.0022	.0031	.0022	.0012
2.0	.0002	.0002	.0002	.0001

TABLE 18.- SHEAR STRESSES τ/τ_o FOR ALL

ELLIPSES FROM $(2\lambda_n b)_o$

(a) $a/b = 0.0871$

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$	$\frac{s}{l} = \frac{1}{4}$
0	3.04	3.14	3.56	4.66	6.35	34.80
1/8	4.29	5.16	5.36	7.68	7.90	24.61
1/4	5.04	5.96	6.20	8.24	7.33	24.53
1/2	5.97	6.87	6.89	8.09	6.34	24.91
3/4	6.48	7.30	7.09	7.88	5.94	25.16
1.0	6.74	7.50	7.15	7.76	5.76	25.54
2.0	6.98	7.69	7.19	7.64	6.72	25.34

(b) $a/b = 0.1738$.

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$	$\frac{s}{l} = \frac{1}{4}$
0	1.92	2.01	2.28	3.12	4.38	10.97
1/8	2.68	2.78	2.97	4.26	6.25	6.32
1/4	3.10	3.14	3.35	4.48	5.99	5.67
1/2	3.55	3.56	3.66	4.39	5.63	5.36
3/4	3.78	3.77	3.75	4.30	5.49	5.29
1.0	3.89	3.86	3.79	4.25	5.44	5.27
2.0	3.99	3.94	3.80	4.21	5.39	4.87

(c) $a/b = 0.2588$.

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$	$\frac{s}{l} = \frac{1}{4}$
0	1.47	1.52	1.75	2.15	3.40	5.69
1/8	1.85	1.90	2.20	2.51	3.52	3.86
1/4	2.09	2.12	2.41	2.56	3.27	3.38
1/2	2.23	2.39	2.60	2.47	3.01	3.94
3/4	2.53	2.51	2.65	2.41	2.90	3.00
1.0	2.60	2.57	2.66	2.37	2.86	2.96
2.0	2.66	2.61	2.67	2.35	2.83	2.93

TABLE 18.- SHEAR STRESSES τ/τ_o FOR ALL

ELLIPSES FROM $(2\lambda_n b)_o$ - Concluded

(d) a/b = 0.5000.

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$	$\frac{s}{l} = \frac{1}{4}$
0	1.12	1.17	1.30	1.68	1.98	2.23
1/8	1.25	1.30	1.42	1.72	1.88	1.96
1/4	1.35	1.40	1.50	1.71	1.80	1.84
1/2	1.48	1.51	1.56	1.68	1.71	1.74
3/4	1.54	1.56	1.59	1.66	1.69	1.70
1.0	1.57	1.59	1.60	1.65	1.67	1.68
2.0	1.60	1.61	1.60	1.64	1.66	1.67

(e) a/b = 0.8660.

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$	$\frac{s}{l} = \frac{1}{4}$
0	1.00	1.02	1.07	1.13	1.14	1.16
1/8	1.02	1.04	1.07	1.12	----	1.13
1/4	1.04	1.05	1.07	1.11	----	1.12
1/2	1.06	1.06	1.07	1.09	----	1.10
3/4	1.07	1.07	1.07	1.09	----	1.08
1.0	1.07	1.07	1.07	1.08	----	1.08
2.0	1.08	1.08	1.07	1.08	----	1.07

TABLE 19.- COORDINATES OF NACA 63_1-012 AIRFOIL

SECTION TORSION BOX

$\frac{s}{l}$	$\frac{r}{l}$	$\frac{s}{l}$	$\frac{r}{l}$ [1]
0	0.259	0	0.2244
.015	.141	.025	.1138
.028	.105	.050	.0689
.047	.084	.075	.0582
.065	.065	.100	.0488
.083	.061	.125	.0488
.119	.050	.150	.0487
.155	.049	.175	.0511
.190	.046	.200	.0477
.225	.043	.225	.0476
.261	.042	.250	.0494
.296	.042	.275	.0466
.332	.042	.300	.0457
.367	.045	.325	.0485
.402	.045	.350	.0447
.437	.047	.375	.0440
.473	.050	.400	.0461
.500	.198	.425	.0430
		.450	.0385
		.475	.1512
		.500	.2160

[1] $\frac{r}{l}$ computed from $\frac{r}{l} = C_o' + \sum C_n' \cos \frac{2\pi ns}{l}$

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TABLE 20.- SERIES COEFFICIENTS FOR

NACA 63_1-012 AIRFOIL SECTION

n	C_n'	B_n	$(2\lambda_n b)$
1	0.00288	-0.00660	2.31
2	.03445	-.04886	4.86
3	-.00205	.00206	7.38
4	.03068	-.01870	9.41
5	-.00386	.00190	11.79
6	.02553	-.01033	14.10
7	-.00409	.00160	16.90
8	.02054	-.00598	18.61
9	-.00254	.00053	21.31
10	.01613	-.00367	23.12
11	-.00044	.00030	26.46
12	.01221	-.00237	27.92
13	.00185	-.00045	32.25
14	.00588	-.00134	33.79
15	.00377	-.00061	35.26
16	.00351	-.00056	38.06
17	.00549	-.00093	40.98
18	.00183	-.00006	43.03
19	.00668	-.00006	44.76
20	-.00038	.00015	48.01

TABLE 21.- ANGLES OF TWIST $\theta/(M/I_oG)$ AND $\theta/(M/IG)$

FOR NACA 63_1-012 AIRFOIL SECTION

$\frac{x}{2b}$	$\frac{\theta}{M/I_oG}$	$\frac{\theta}{M/IG}$
0	8.70	1.00
1/4	11.64	1.34
1/2	12.33	1.42
1.0	12.61	1.45
1.5	12.67	1.46
2.0	12.69	1.46
4.0	12.69	1.46

TABLE 22.- SUMMARY OF NORMAL STRESSES σ_x' FOR

NACA 63_1-012 AIRFOIL SECTION

$\frac{x}{2b}$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$	$\frac{s}{l} = \frac{1}{4}$	$\frac{s}{l} = \frac{3}{8}$	$\frac{s}{l} = \frac{7}{16}$	$\frac{s}{l} = \frac{15}{32}$
0	-1.90	-1.20	-0.83	-0.18	-0.48	0.83	2.38	5.13
1/4	-1.00	-.82	-.47	-.40	-.11	.70	1.07	.72
1/2	-.36	-.39	-.25	-.16	-.07	.32	.35	.22
1.0	-.09	-.12	-.09	-.07	-.03	.08	.07	.04
1.5	-.02	-.03	-.03	-.02	-.01	.01	.01	.01
2.0	-.01	-.01	-.01	-.01	-.01	0	0	0
4.0	0	0	0	0	0	0	0	0

TABLE 23.- SUMMARY OF SHEAR STRESSES τ/τ_o FOR

NACA 63_1-012 AIRFOIL SECTION

$\frac{x}{2b}$	$\frac{s}{l} = 0$	$\frac{s}{l} = \frac{1}{16}$	$\frac{s}{l} = \frac{1}{8}$	$\frac{s}{l} = \frac{3}{16}$	$\frac{s}{l} = \frac{7}{32}$	$\frac{s}{l} = \frac{1}{4}$	$\frac{s}{l} = \frac{3}{8}$	$\frac{s}{l} = \frac{7}{16}$	$\frac{s}{l} = \frac{15}{32}$	$\frac{s}{l} = \frac{1}{2}$
0	9.85	2.62	1.90	1.71	1.63	1.60	1.71	1.66	1.90	7.53
1/4	7.48	3.95	2.26	1.96	2.23	2.68	3.77	3.97	.04	2.91
1/2	6.32	1.80	3.18	3.17	3.11	3.21	3.63	4.21	.03	3.10
1.0	6.19	1.64	3.30	3.43	3.38	3.49	3.82	4.09	-.24	2.92
1.5	6.14	1.57	3.31	3.50	3.46	3.60	3.85	4.05	-.31	2.89
2.0	6.13	1.56	3.30	3.52	3.48	3.62	3.87	4.05	-.32	2.89
4.0	6.12	1.55	3.29	3.52	3.49	3.63	3.88	4.06	-.32	2.89

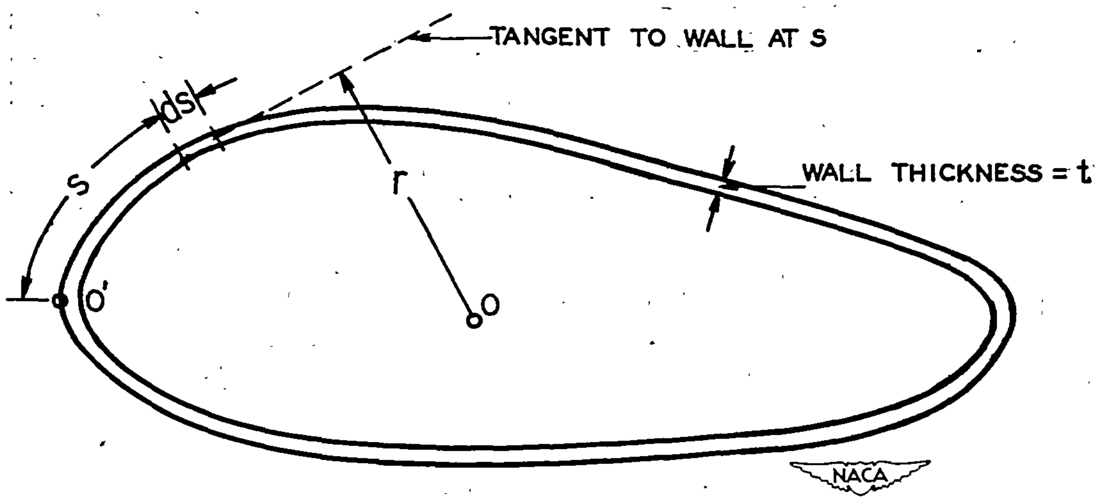


Figure 1.- Cross section of torsion tube.

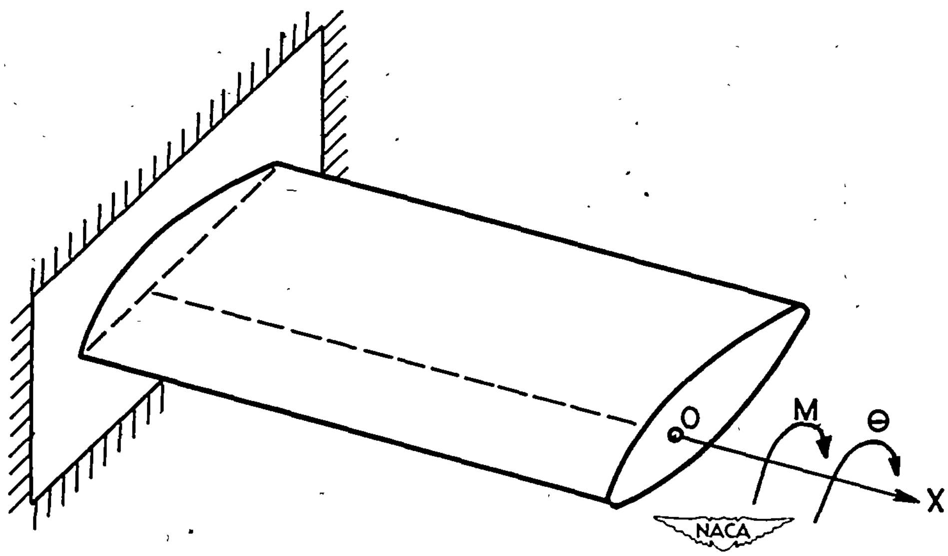


Figure 2.- Perspective of torsion tube.

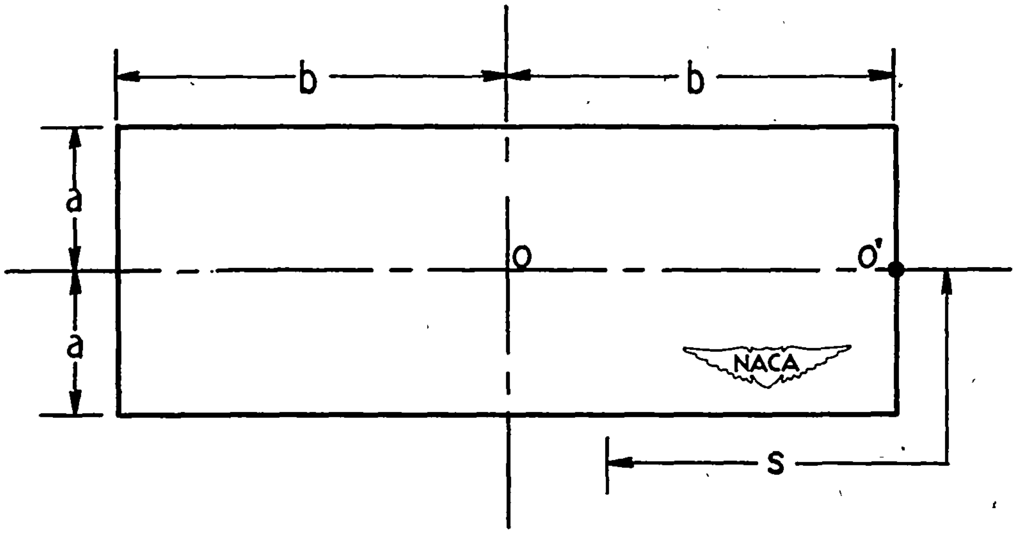


Figure 3.- Notation for rectangular box.

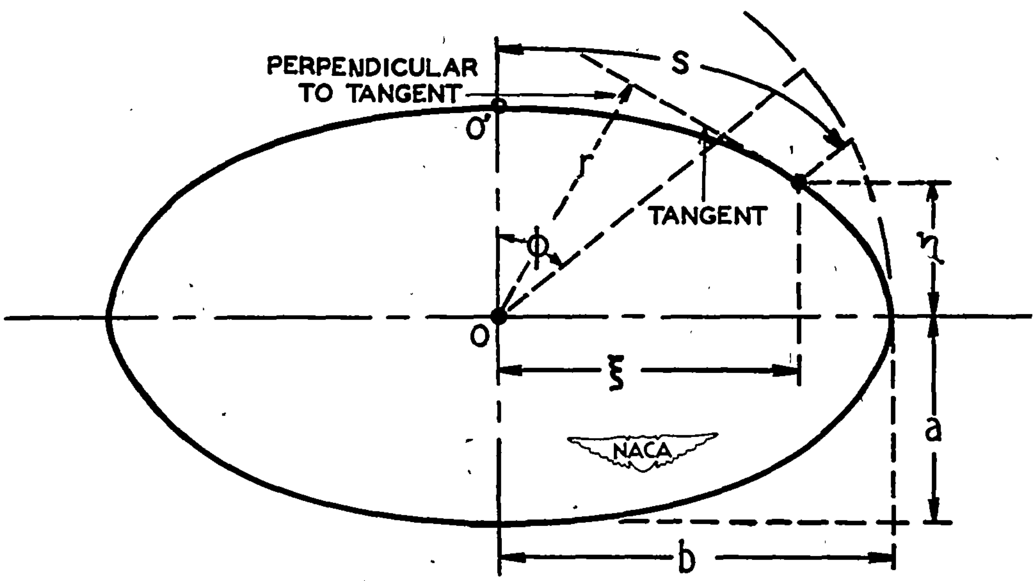


Figure 4.- Notation for elliptical box.

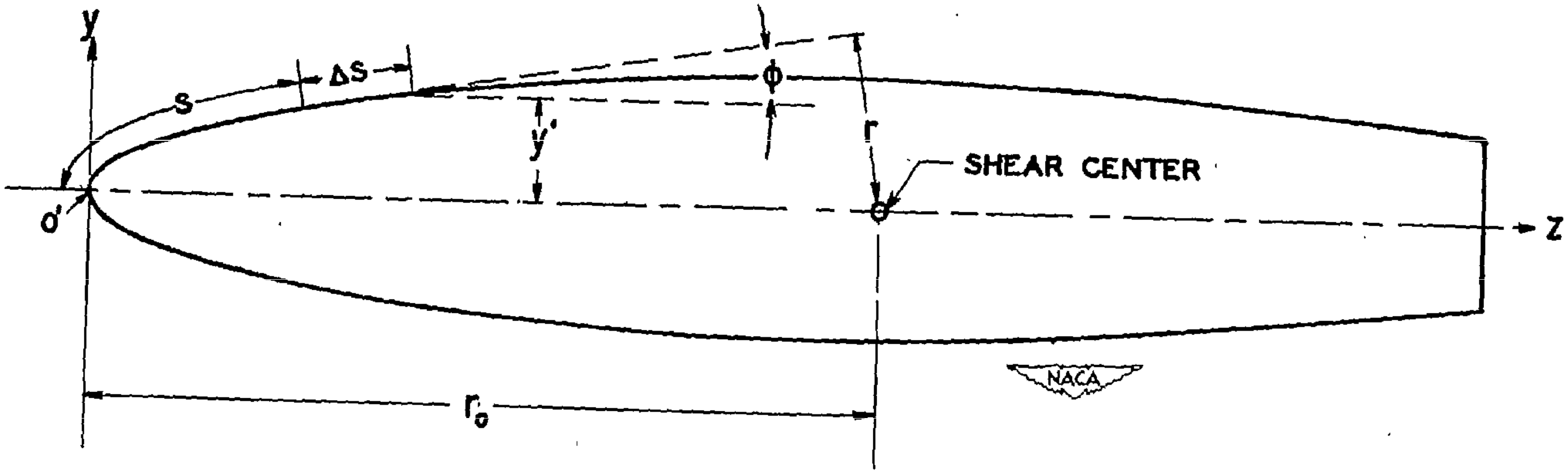
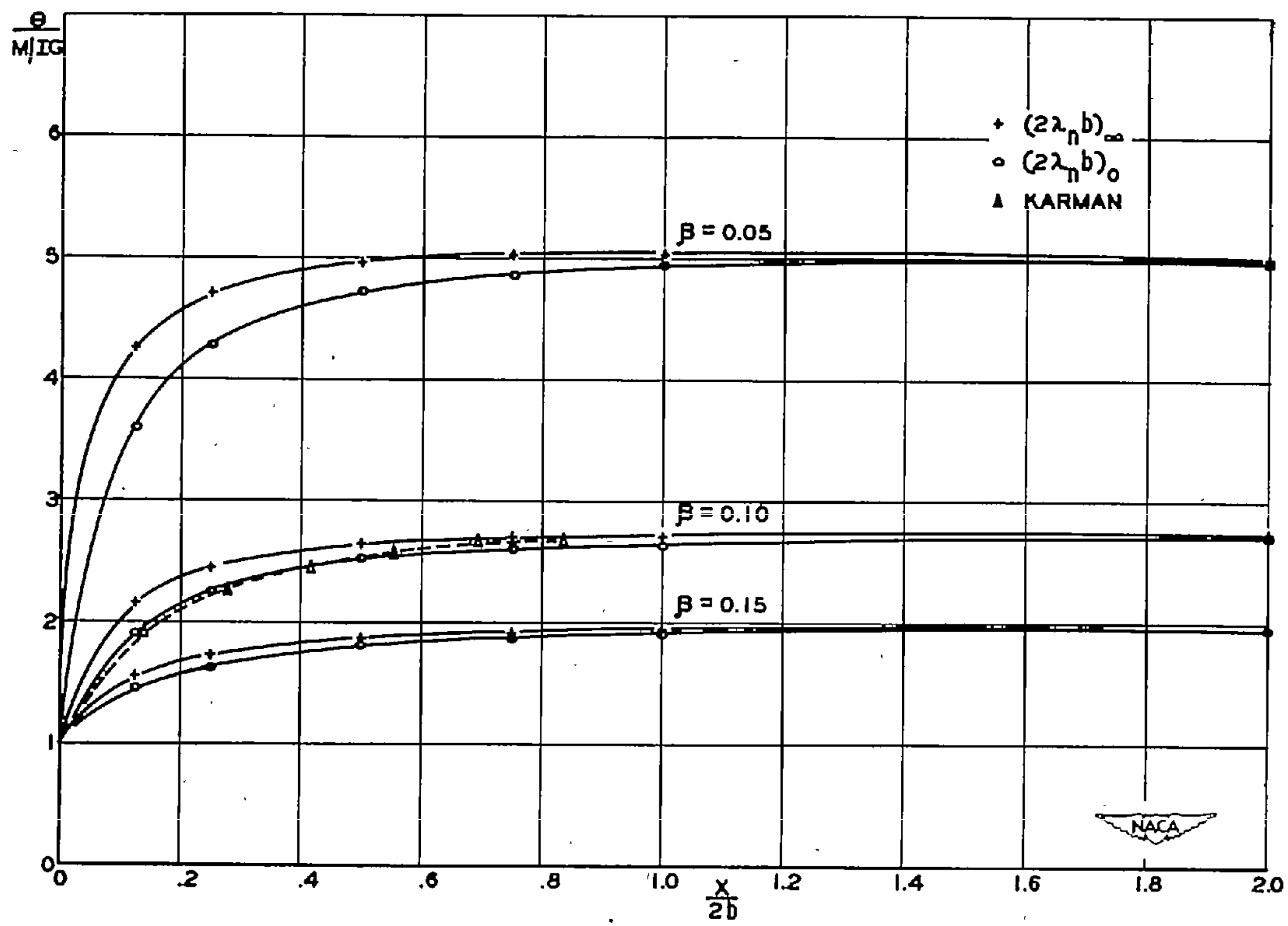
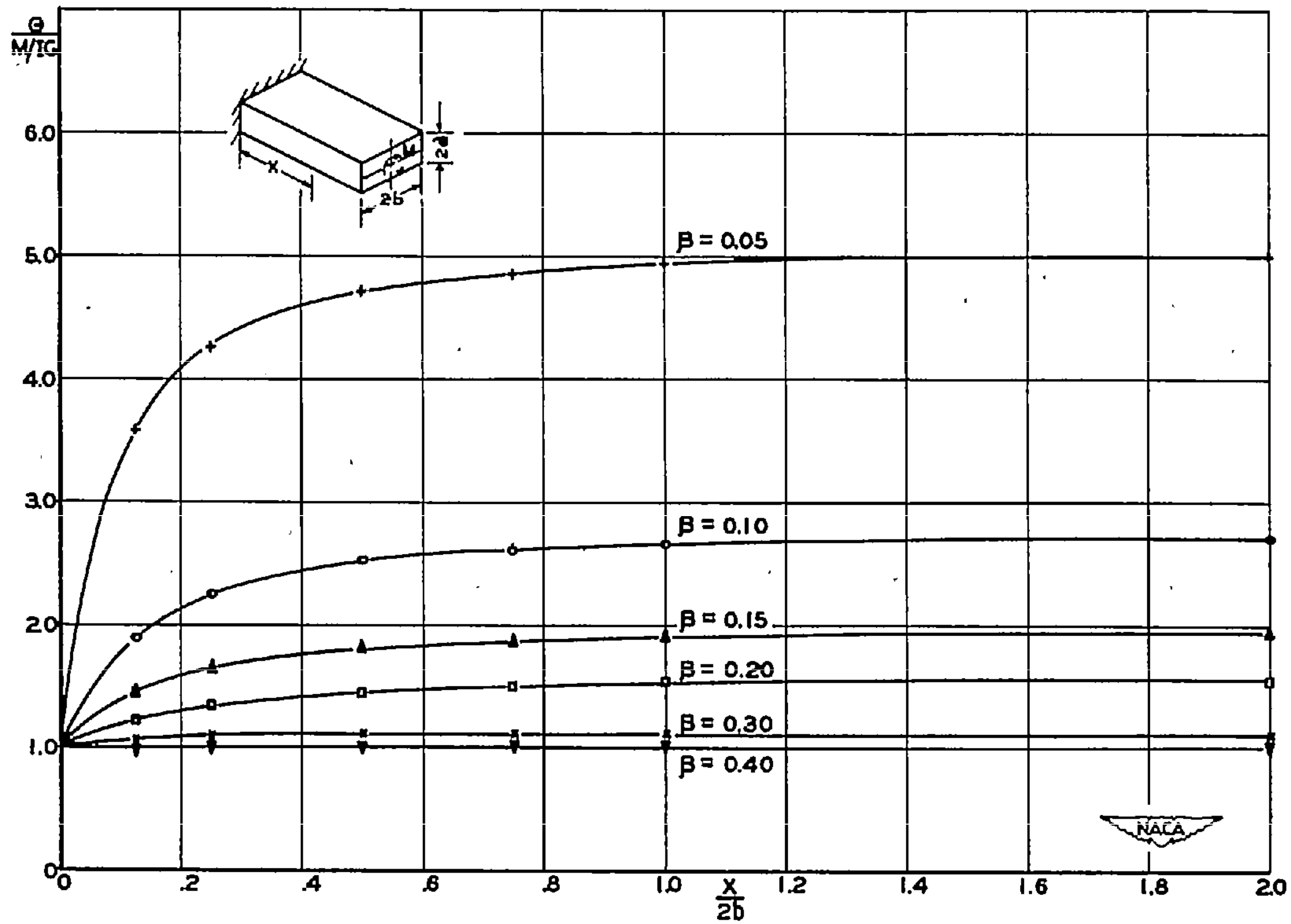


Figure 5.- Torsion tube 65 percent of NACA 63_1-012 airfoil section.



(a) Values based on λ_0 and λ_∞. Comparison with values obtained by Kármán and Chien (reference 1) is shown for $\beta = 0.10$.

Figure 6.- Variation of angles of twist $\theta(M/IG)$ with $x/2b$ for rectangular boxes with various β values.



(b) Values based on λ_o.

Figure 6.- Concluded.

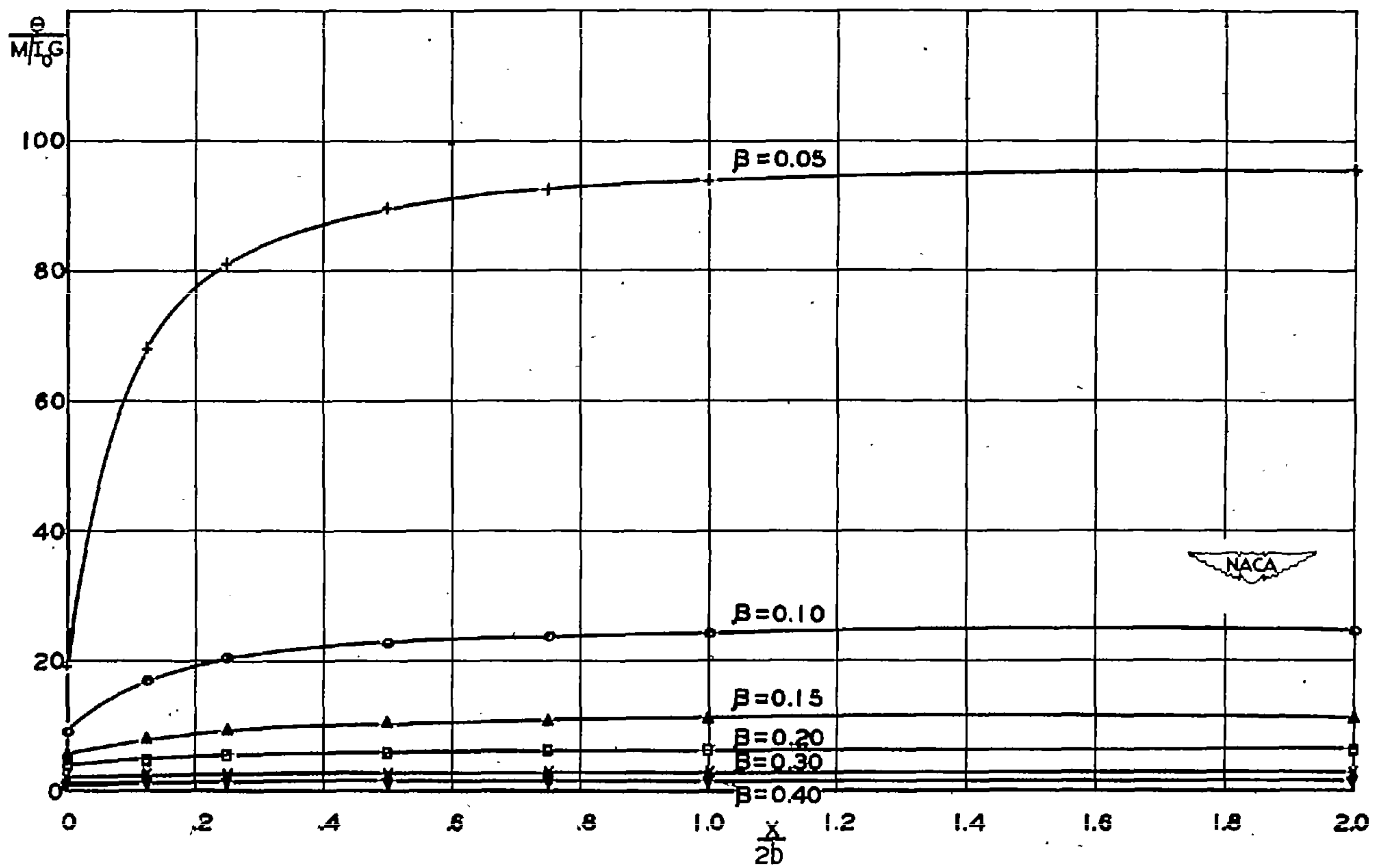


Figure 7.- Variation of angles of twist $\theta/(MI_0G)$ with $x/2b$ for rectangular boxes with various β values and based on λ_0.

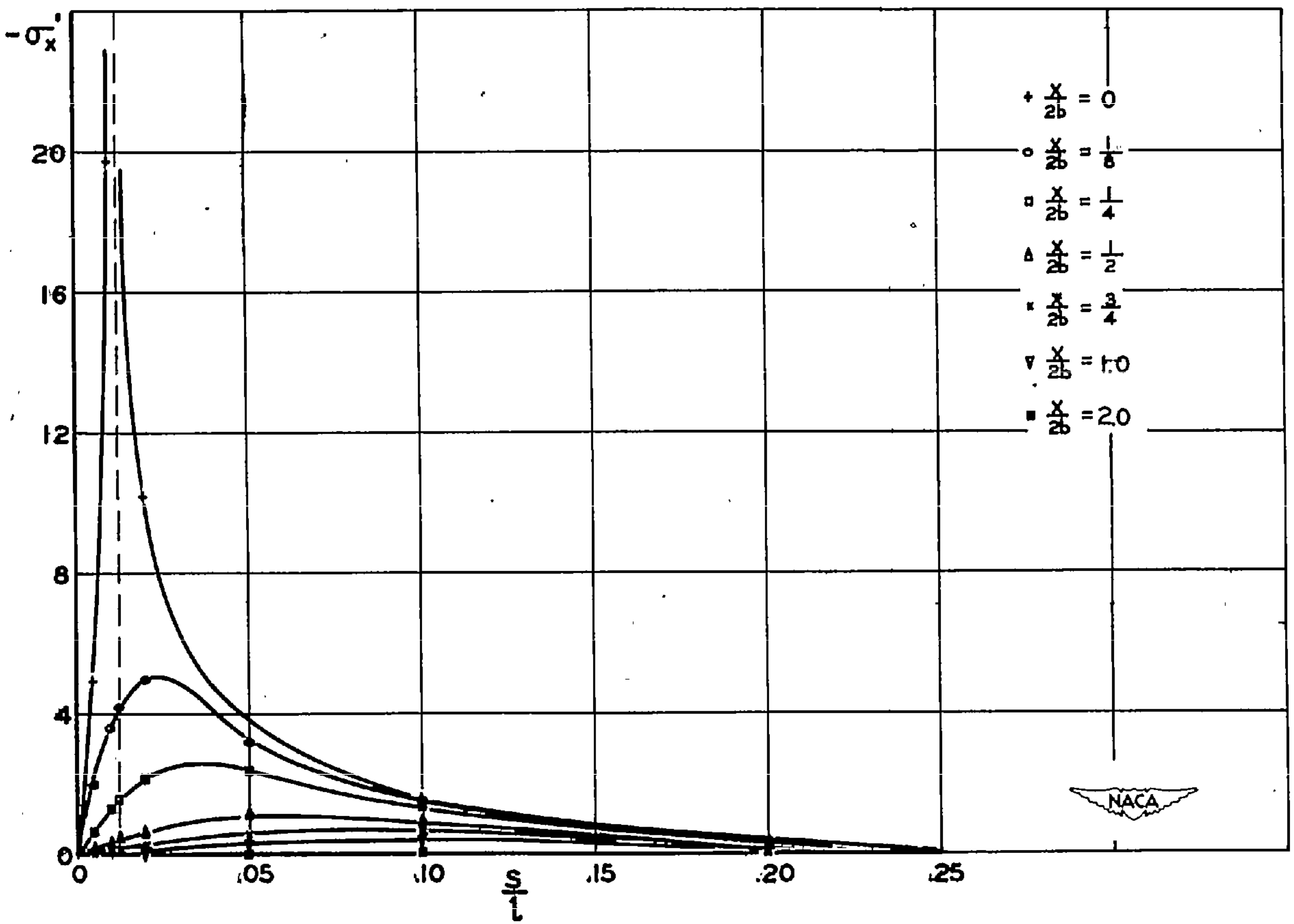


Figure 8.- Variation of normal stresses $-\sigma_x'$ with s/l for rectangular boxes with $\beta = 0.05$.

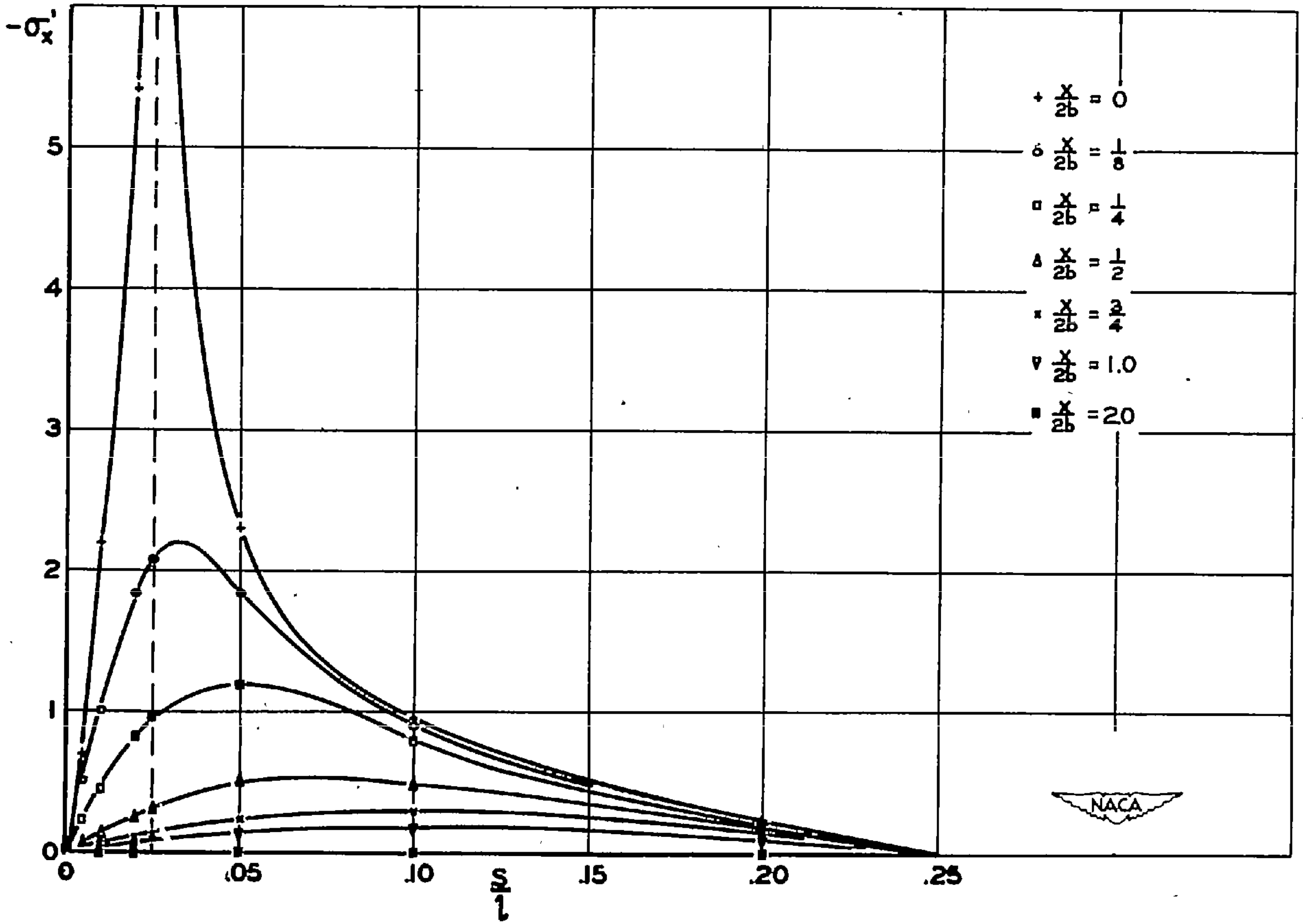


Figure 9.- Variation of normal stresses $-\sigma_x'$ with s/l for rectangular boxes with $\beta = 0.10$.

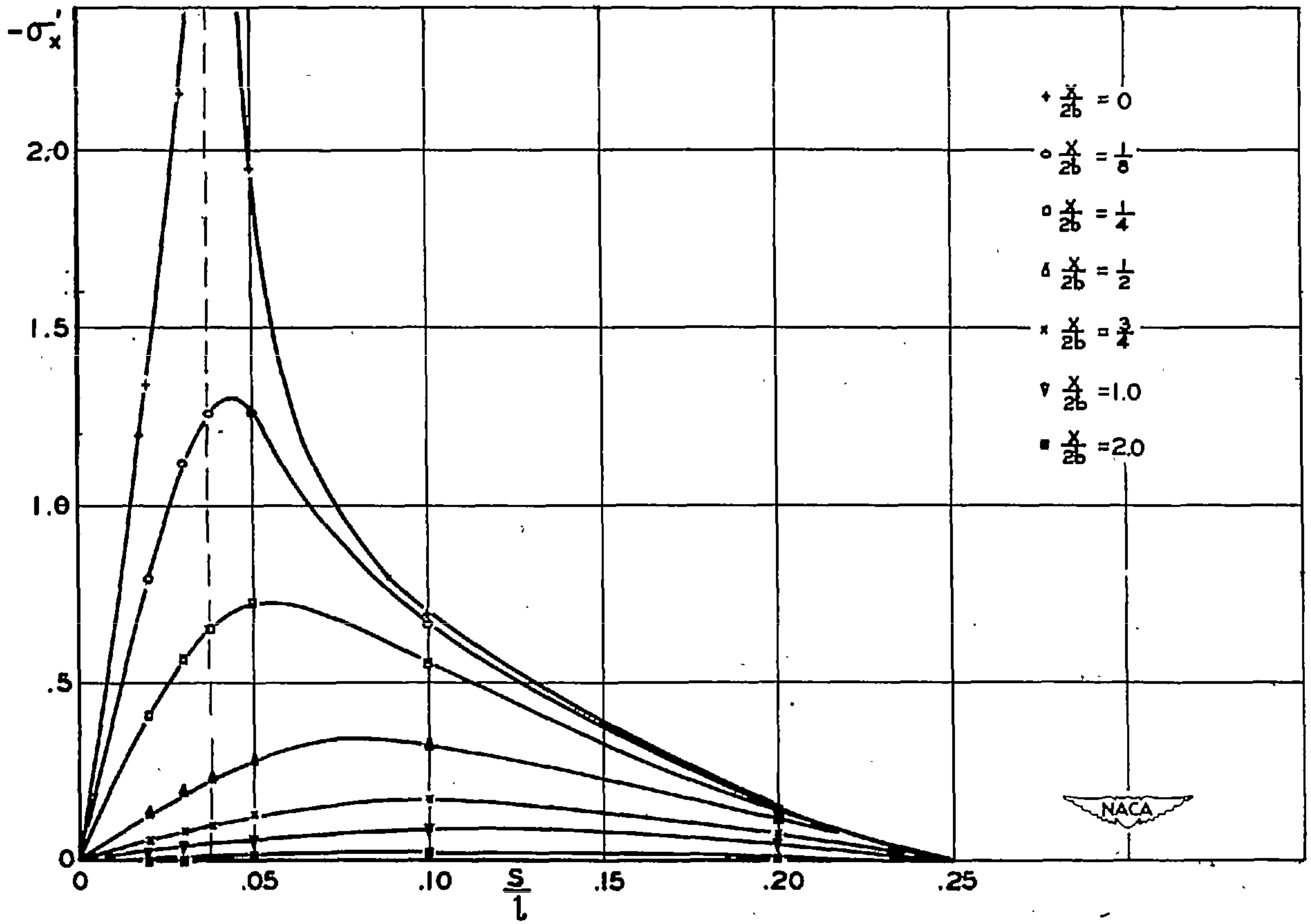


Figure 10.- Variation of normal stresses $-\sigma_x'$ with s/l for rectangular boxes with $\beta = 0.15$.

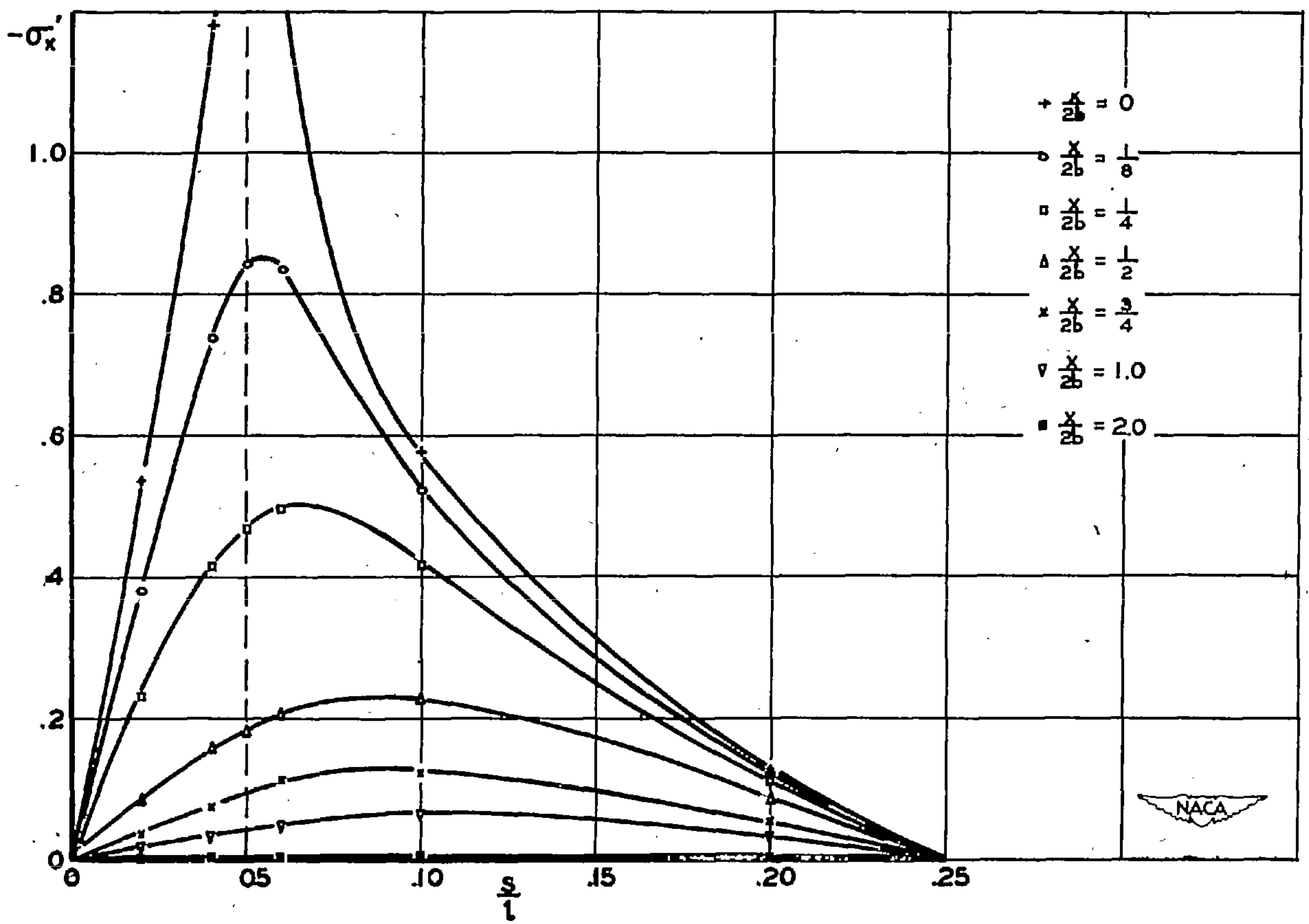


Figure 11.- Variation of normal stresses $-\sigma_x'$ with s/l for rectangular boxes with $\beta = 0.20$.

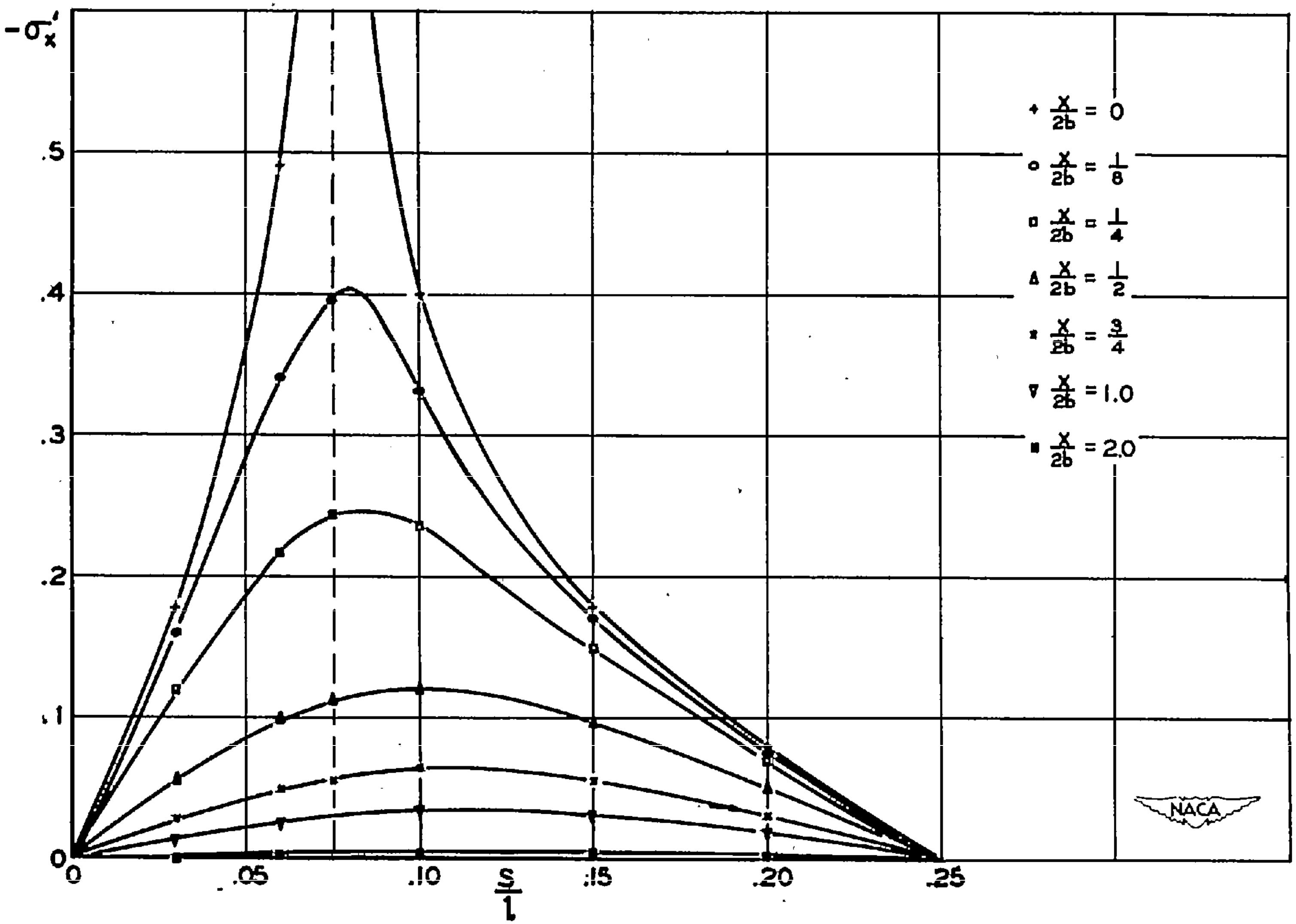


Figure 12.- Variation of normal stresses $-\sigma_x'$ with s/l for rectangular boxes with $\beta = 0.30$.

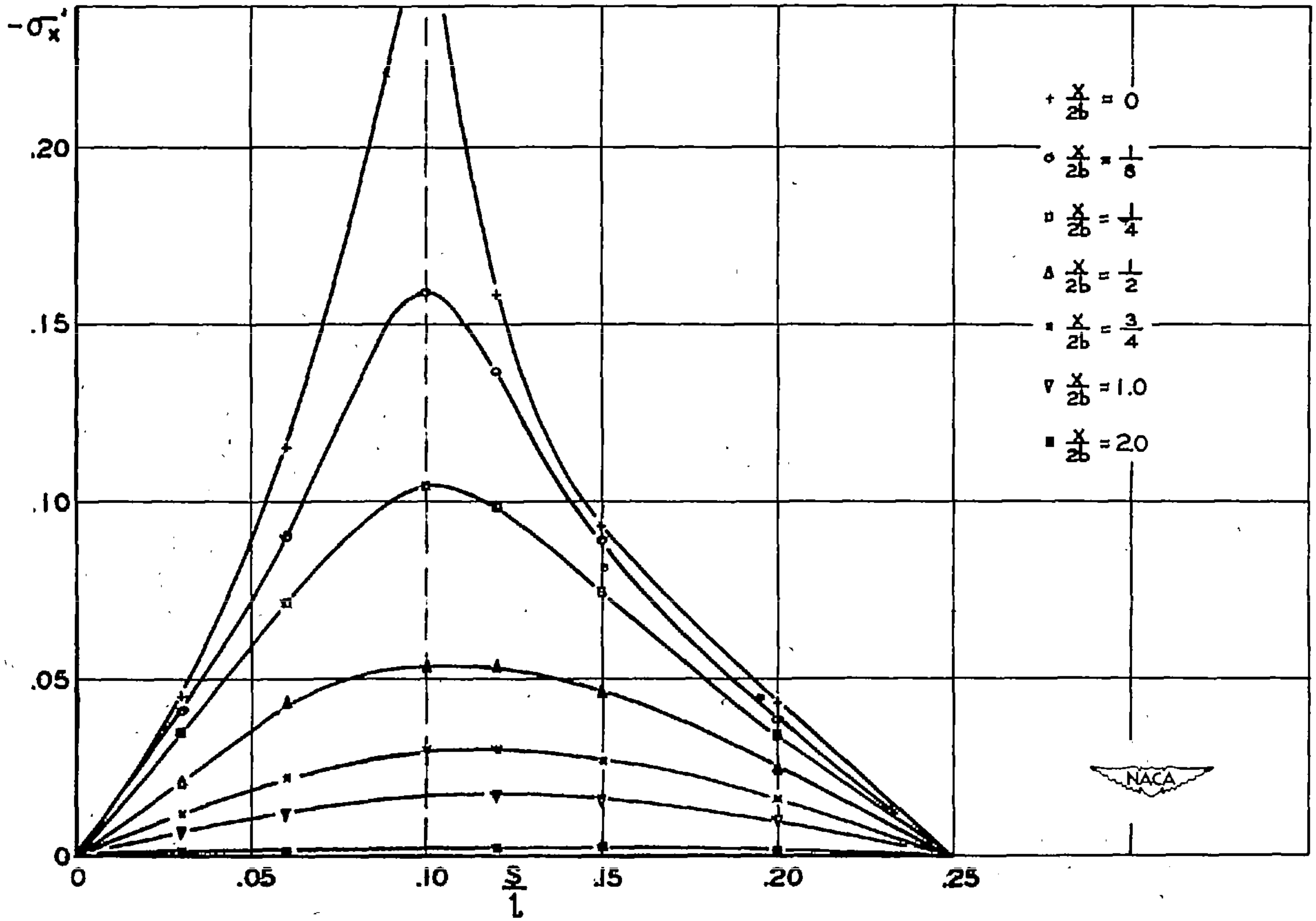


Figure 13.- Variation of normal stresses $-\sigma_x'$ with s/l for rectangular boxes with $\beta = 0.40$.

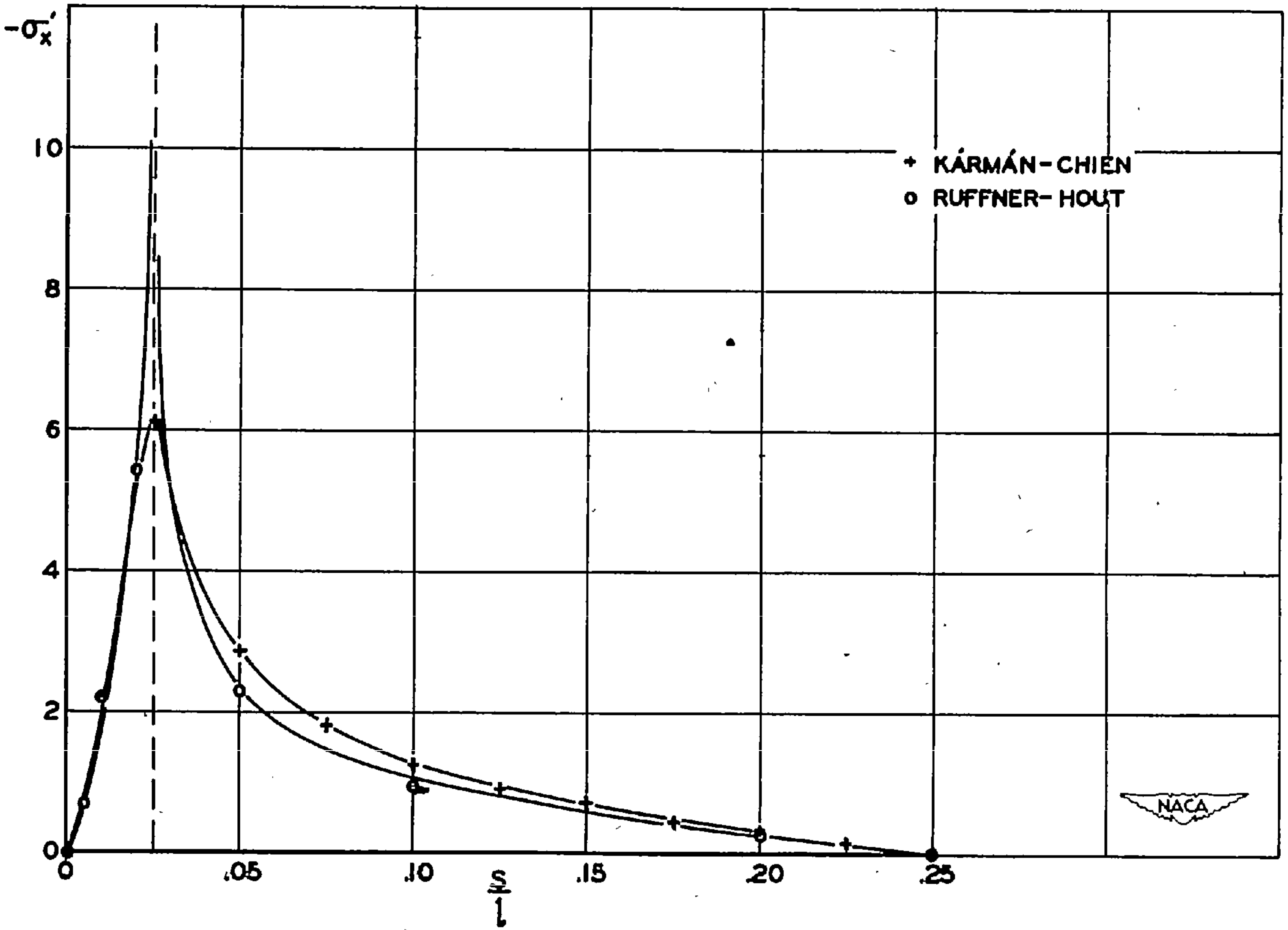


Figure 14.- Comparison of normal stresses for rectangular box with $\beta = 0.1$. $-\sigma_x'$ at $x/2b = 0$.

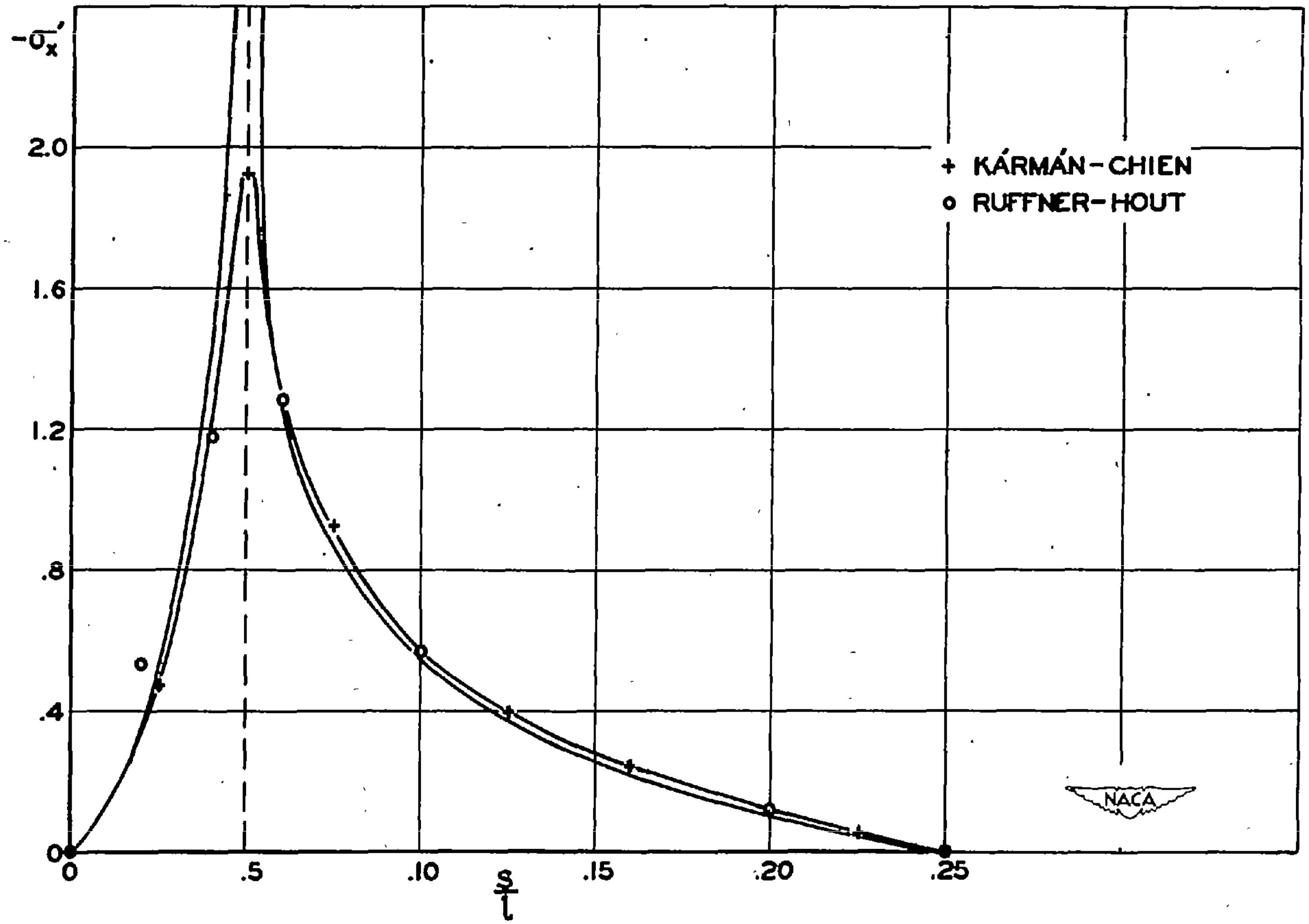


Figure 15.- Comparison of normal stresses for rectangular box with $\beta = 0.2$. $-\sigma_x'$ at $x/2b = 0$.

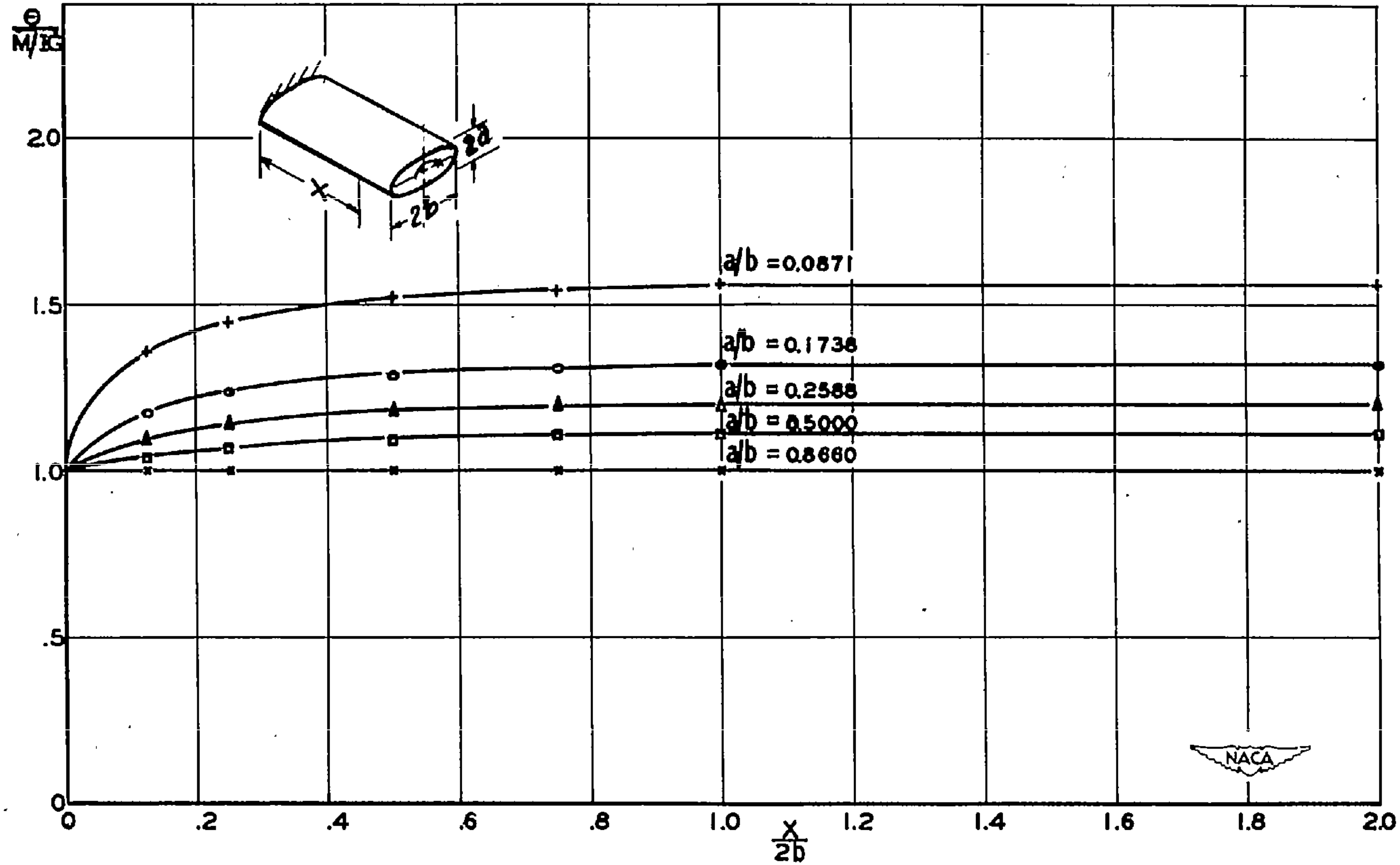


Figure 16.- Variation of angles of twist $\theta/(M/IG)$ with $x/2b$ for elliptical boxes

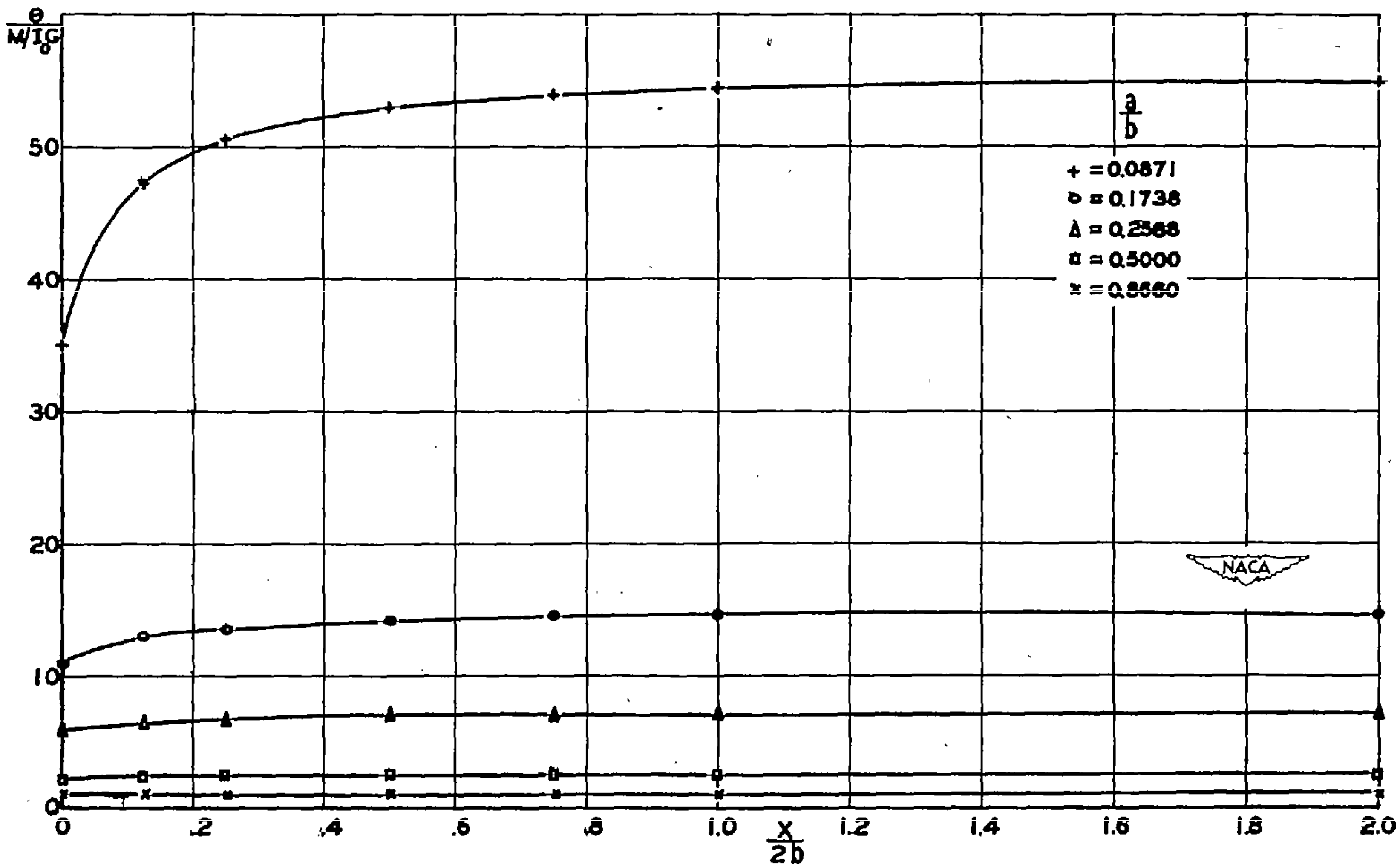


Figure 17.- Variation of angles of twist $\theta/(M/I_oG)$ with $x/2b$ for elliptical boxes.

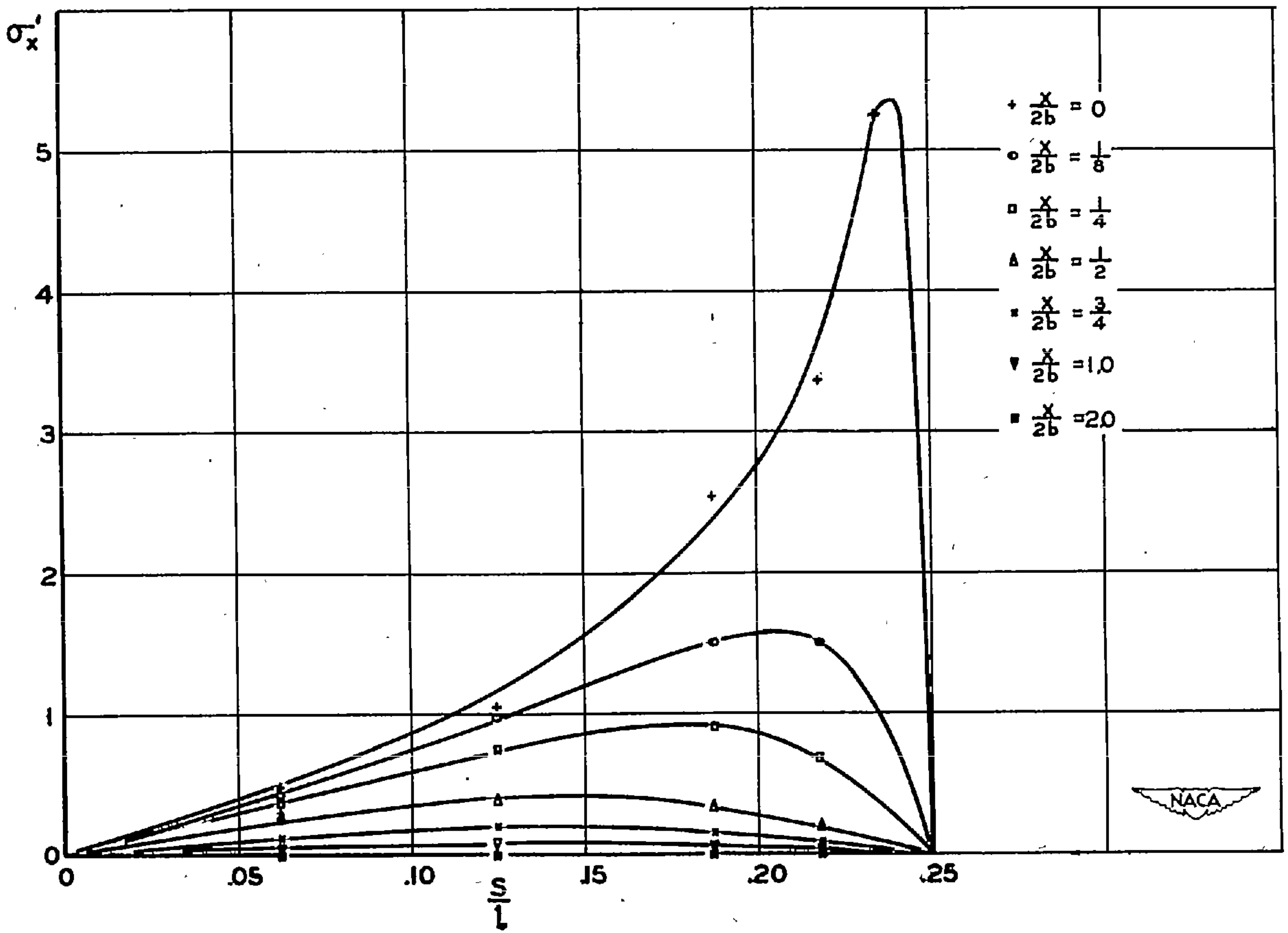


Figure 18.- Variation of normal stresses σ_x' with s/l for elliptical boxes with $a/b = 0.0871$.

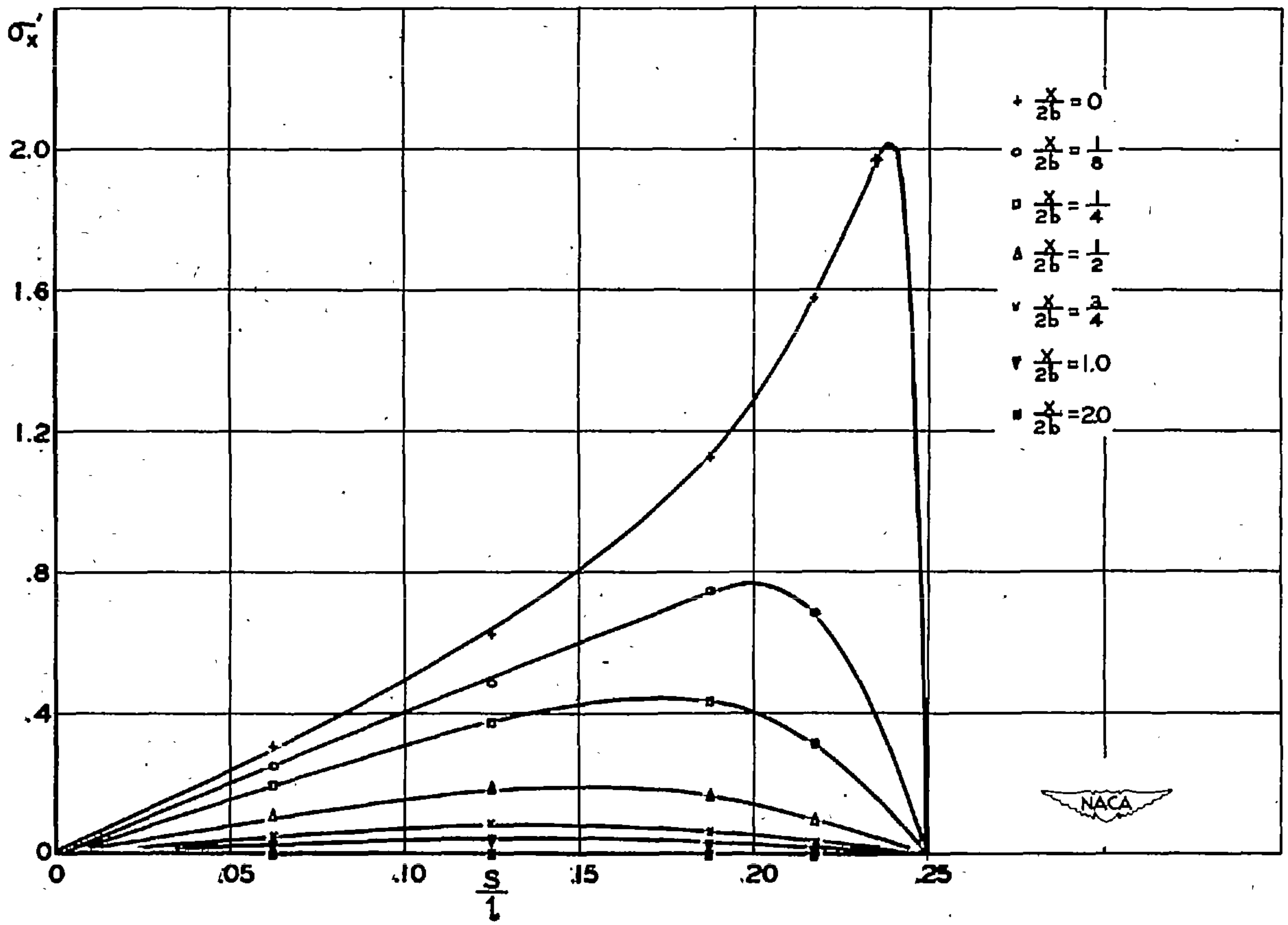


Figure 19.- Variation of normal stresses σ_x' with s/l for elliptical boxes with $a/b = 0.1738$.

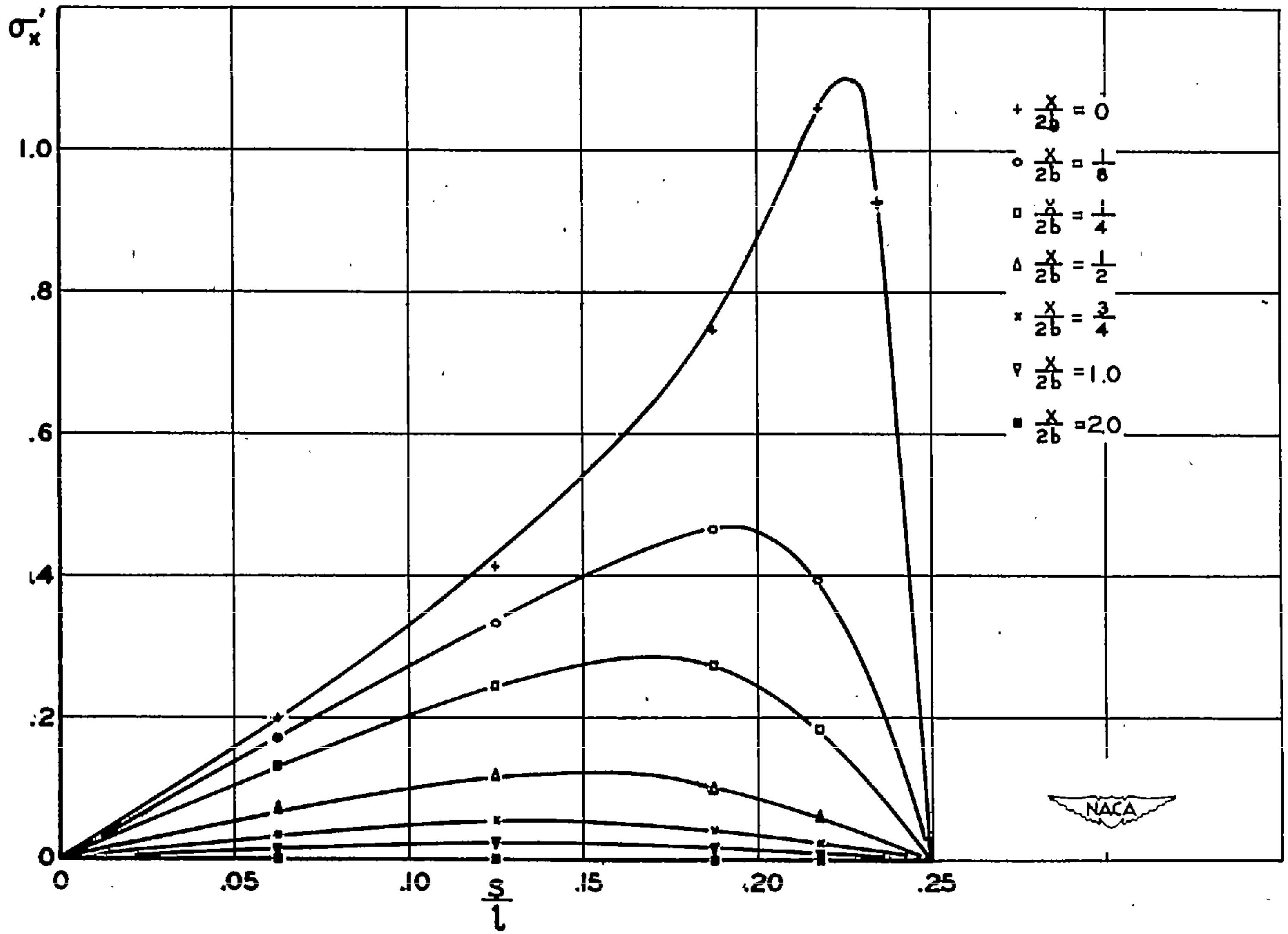


Figure 20.- Variation of normal stresses σ_x' with s/l for elliptical boxes with $a/b = 0.2588$.

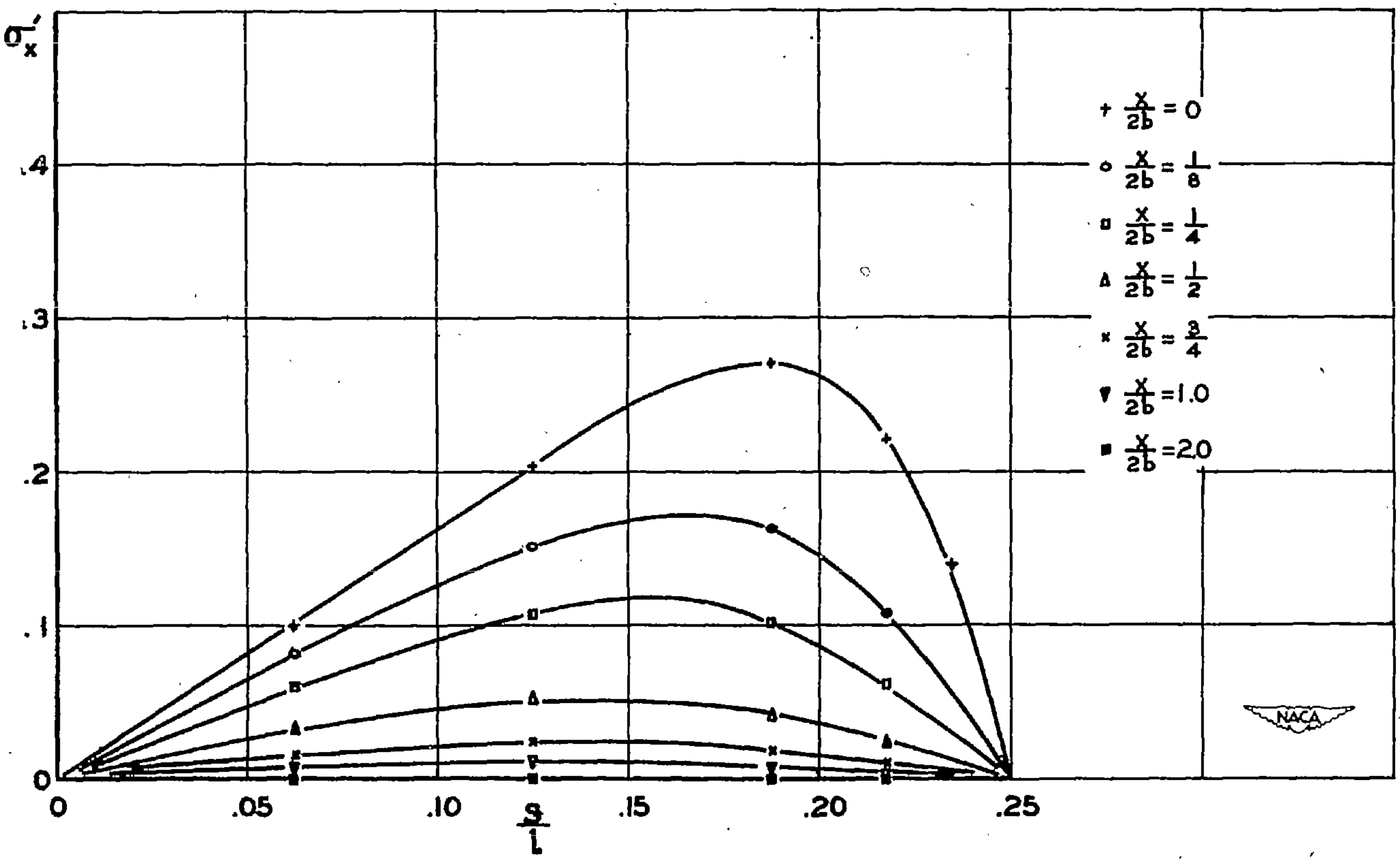


Figure 21.- Variation of normal stresses σ_x' with s/l for elliptical boxes with $a/b = 0.5000$.

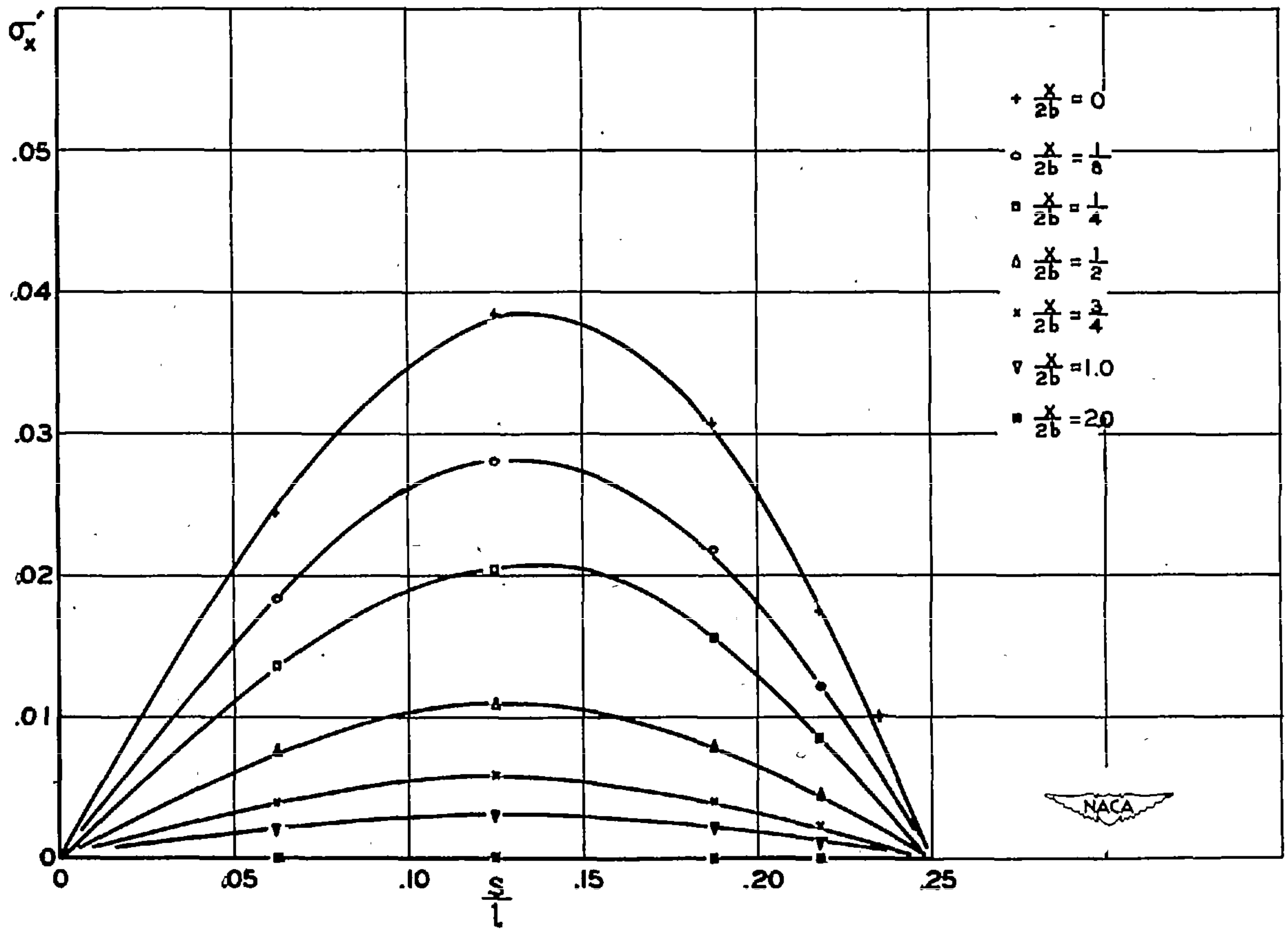


Figure 22.- Variation of normal stresses σ_x' with s/l for elliptical boxes with $a/b = 0.8660$.

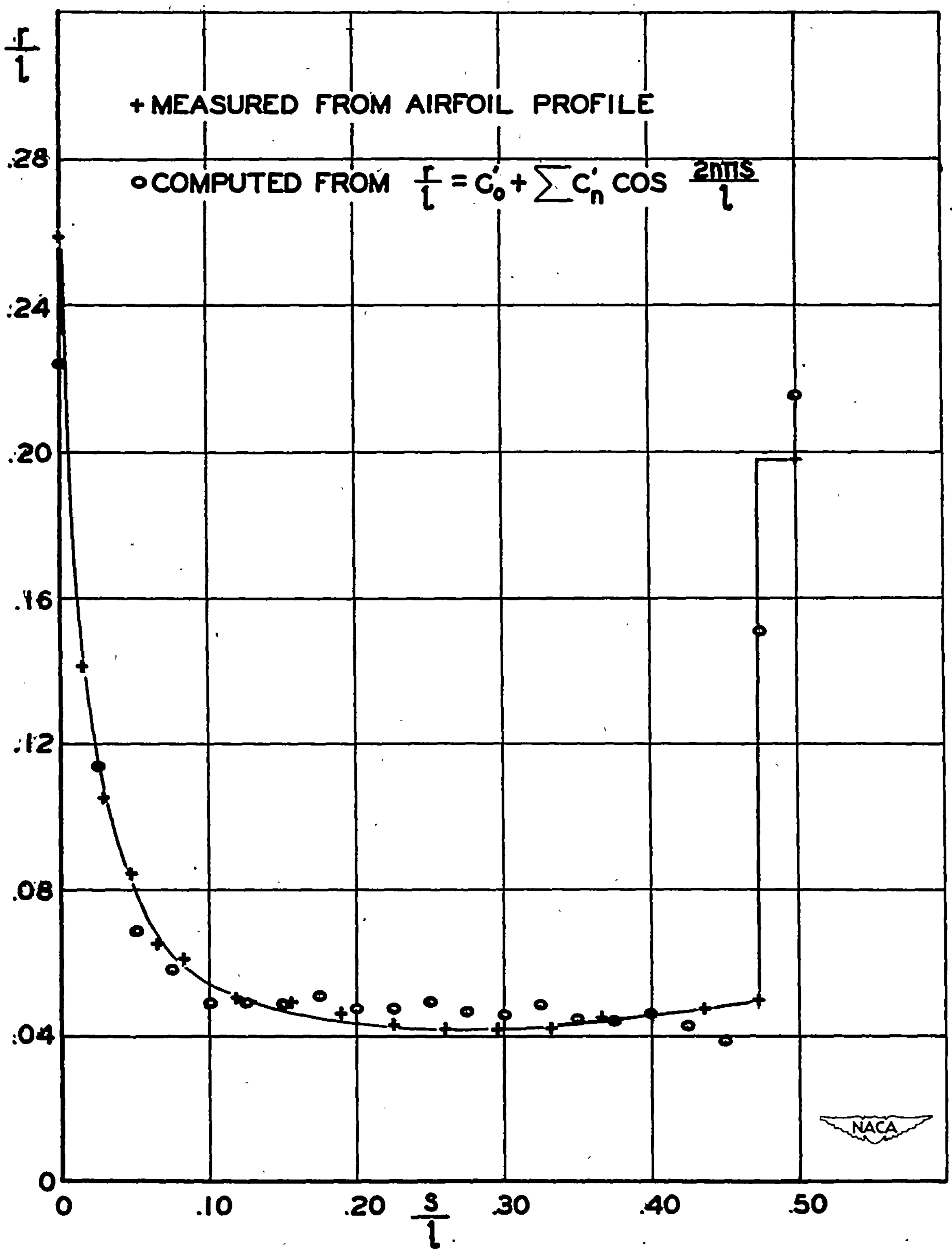


Figure 23.- Contour of NACA 63_1-012 airfoil.

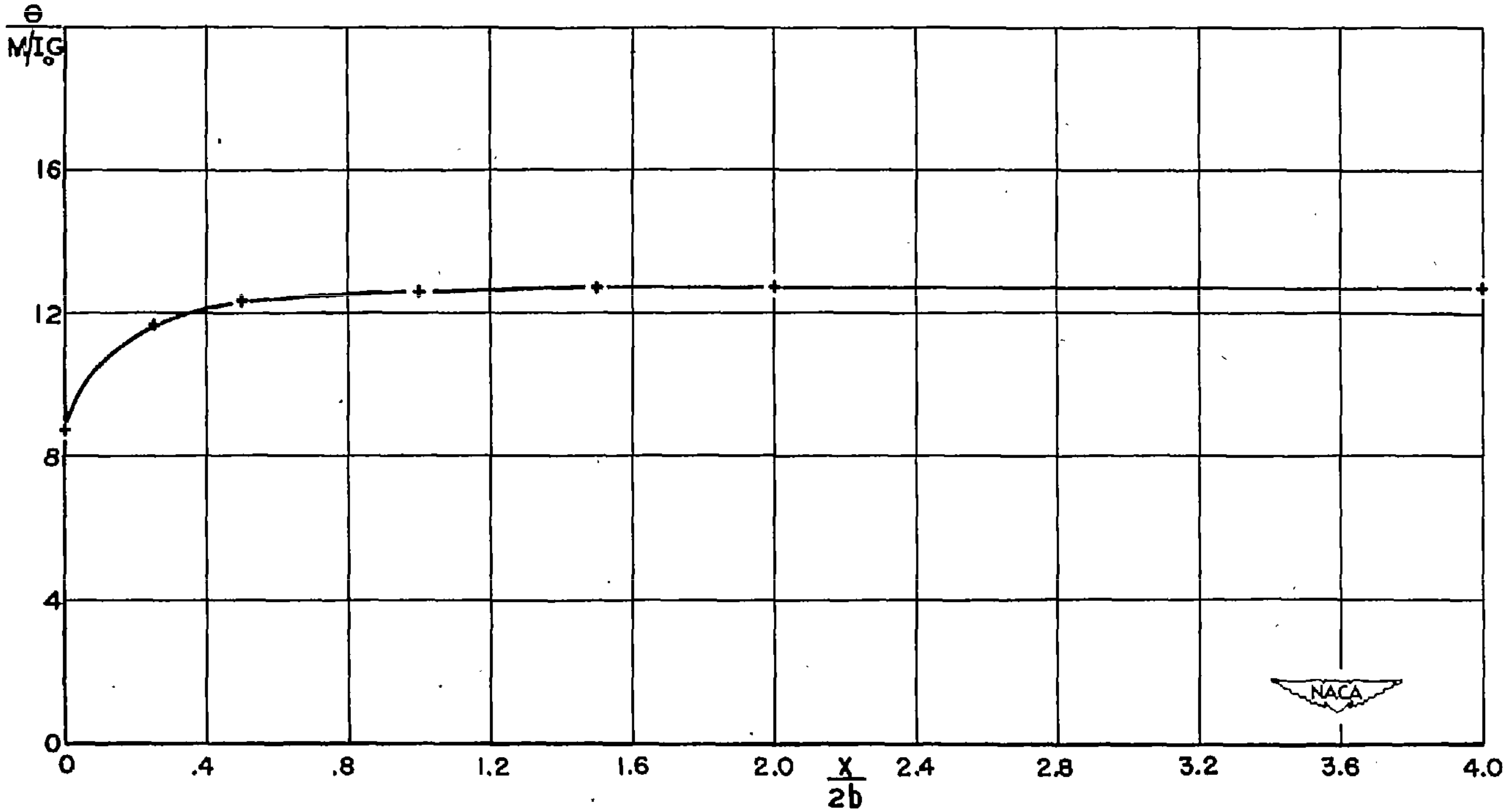


Figure 24.- Variation of angles of twist $\theta/(M/I_oG)$ with $x/2b$ for NACA 63_1-012 airfoil section.

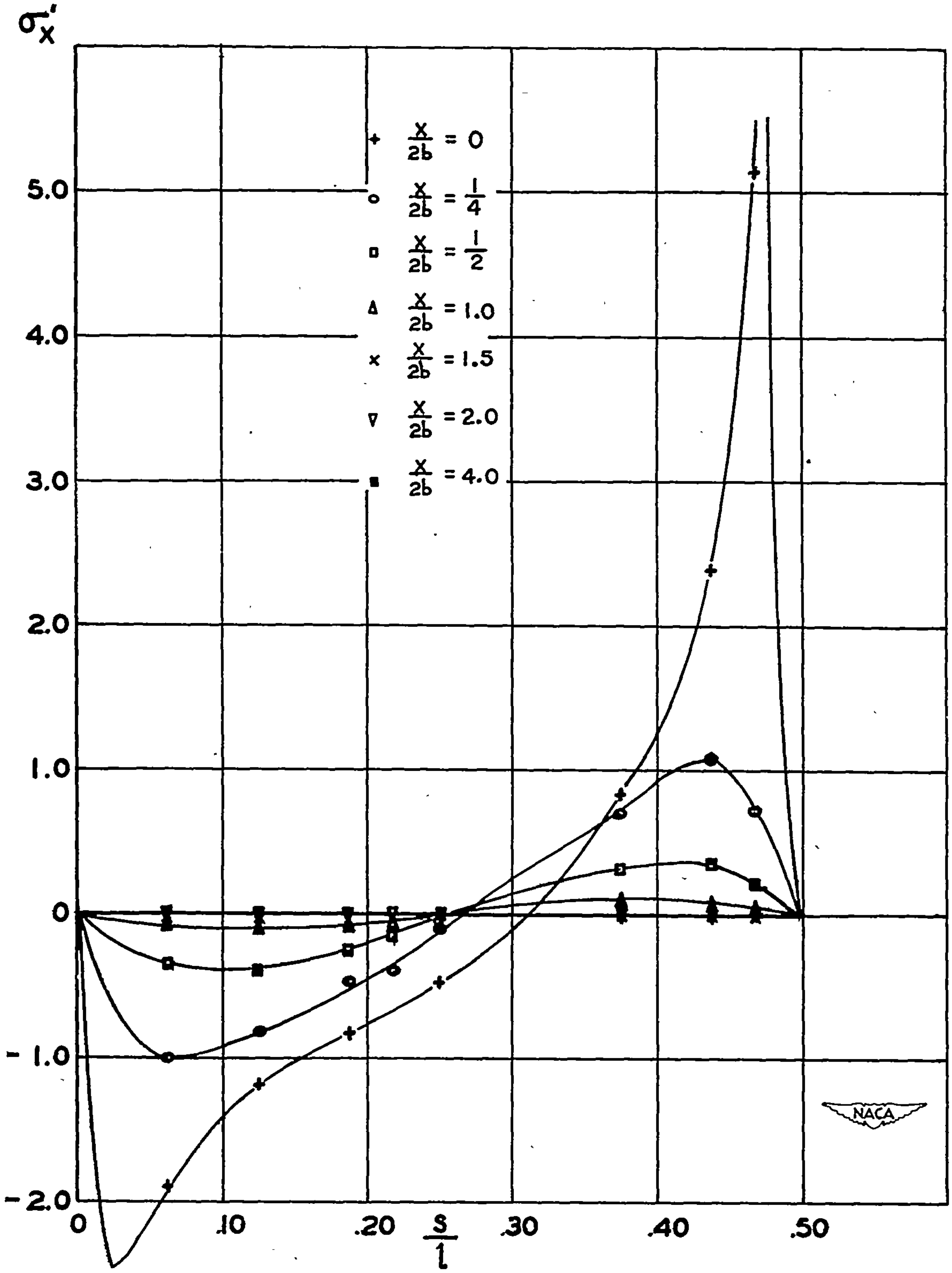


Figure 25.- Variation of normal stresses σ_x' with s/l for NACA 63_1-012 airfoil.